Smallest Enclosing Ball for a Point Set with Strictly Convex Level Sets

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Abstract

Let the center point be the point that minimizes the maximum distance from a point of a given point set to the center point. Finding this center point is referred to as the *smallest enclosing ball problem.* In case of points with Euclidean distance functions, the smallest enclosing ball is actually the center of a geometrical ball.

We consider point sets with points that have distance functions with strictly convex level sets. For such point sets, we show that the smallest enclosing ball exists, is unique, and can be computed using an algorithm for solving *LP-type* problems as it was introduced by Sharir and Welzl in [34]. We provide a list of distance functions, show that they have strictly convex level sets, and hint at the implementations of the basic operations used by the LP-type algorithm. In a last part, we analyze *approximative* solutions of the smallest enclosing ball problem and conclude that there are no ϵ*-core sets* for some of the considered distance functions.

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Contents

1 Introduction

We consider the problem of finding the *smallest enclosing ball* for a point set which is defined as finding a center c and a radius r so that the maximum distance r from a point of the point set to the center c is minimized. The distance between points and possible centers can be defined by arbitrary distance functions and dependent on the distance functions, the smallest enclosing ball has different properties; e.g. for some distance functions, the smallest enclosing ball is unique and for others it is not.

The smallest enclosing ball problem has been investigated for several distance functions, e.g. for the Euclidean norm [37] or for *Bregman divergences* [31]. In Section 1.1, we give an overview of the distance functions that have been considered in the literature.

In this thesis, we investigate the smallest enclosing ball problem for a family of distance functions that have strictly convex level sets; a more formal definition will be provided in Section 1.2. Every point from the point set can have its *own* distance function but the distance function has to be from the family of distance functions with strictly convex level sets. The problem of the smallest enclosing ball is reformulated as finding a center c and a radius r so that the maximum distance r — *as perceived by the point* p — between c and a point p from the point set is minimized.

The introduction is structured as follows. First, we give on overview of related areas. Second, we defined the conventions that will be used in this thesis. Third, we acknowledge received support and give an outline. Last, we give an illustration that can be used to present the smallest enclosing ball problem considered in this thesis to *Jane and John Doe.*

1.1 Overview of Related Areas

In this section, we present smallest enclosing ball problems that are related to the setting we are considering here.

1.1.1 Euclidean Norm and Ellipsoids

The smallest enclosing ball problem was according to Nielsen and Nock[1] [30] first mentioned by J.J. Sylvester in 1857 [35]. Let us define the smallest enclosing ball problem for the Euclidean norm:

Definition 1.1. *For a point set* $P \subset \mathbb{R}^d$*, the smallest enclosing ball with*

[1]we use Nielsen's and Nock's overview as a basis for our overview

center c and radius r is a solution of the optimization problem

$$\begin{array}{ll} \textit{minimize} & r \\ \textit{subject to} & \forall p \in P: \ \|c - p\| \leq r \\ & c \in \mathbb{R}^d. \end{array}$$

This definition of the smallest enclosing ball is in accordance with our natural understanding of a smallest enclosing ball; *a balloon of spherical form that is wrapped around points and that is shrunk until it cannot be shrunk further without loosing one of the points*.

Megiddo presented in [28] the first algorithm that solves the smallest enclosing ball problem in linear time for fixed dimension. Welzl presented in [37] a *randomized* algorithm that solves the problem in *expected* linear time for fixed dimension. Welzl's algorithm is more practical as the hidden constants that depend on the dimension are smaller than in Megiddo's algorithm. Further, Fischer, Gärtner, and Kutz present in [14] an algorithm for the smallest enclosing ball problem that performs well in practice for reasonably high dimension.

In [37], Welzl shows that the *smallest enclosing ellipsoid* can be computed in expected linear time for fixed dimension using the same type of algorithm as is applicable to the smallest enclosing ball problem.

1.1.2 LP-Type Problems

As a generalization of the smallest enclosing ball problem as presented in [37], Sharir and Welzl introduced in [34] an abstract framework for problems nowadays called *LP-type problems*. This abstract framework contains the smallest enclosing ball problem but allows a formulation of the problem that is not as close to the geometric properties of a ball as the algorithm presented in [37]. Matoušek, Sharir, and Welzl provide in [27] an improved analysis of the algorithm presented in [34]; they show that the algorithm solves LP-type problems in expected time linear in the number of points and subexponential in the dimension.

As we will see, the problem of the smallest enclosing ball for distance functions with strictly convex level sets is an LP-type problem.

1.1.3 Bregman Balls

Nielsen and Nock claim in [31] that Welzl's algorithm from [37] can be applied to point sets with *Bregman divergences* as they were introduced by Bregman in [3]. The family of Bregman divergences includes the L_2^2 norm, the *Kullback–Leibler divergence* [22], and the *Itakura–Saito divergence*. In general, the triangle inequality does *not* hold for Bregman divergences.

We will show that the problem of finding the smallest enclosing Bregman ball as it was introduced in [31] is part of the family of distance functions that we consider in this thesis and that Welzl's algorithm can indeed be applied.

1.1.4 Smallest Enclosing Ball of Balls

In [29], Megiddo shows that the smallest enclosing ball for a set of *balls* can be computed in linear time for fixed dimension though this is merely a theoretical result. Fischer and Gärtner investigated this problem further, showed in [13] that the smallest enclosing ball of balls problem is an LP-type problem, and provided an implementation.

We will show that the problem of finding the smallest enclosing ball of a set of balls belongs to the family of distance functions that we consider in this thesis. This is not a surprise as we have gotten a substantial part of the ideas used in this thesis from Fischer's thesis.

1.1.5 Smallest Enclosing Cones

For a set of vectors, we define the *smallest enclosing cone* as the cone with smallest angle that contains all vectors. This problem was first considered by Lawson in [24] and he claimed that in $\mathbb{R}^3$, the smallest enclosing ball and the smallest enclosing cone problem are equivalent though he did not provide a proof. Barequet and Elber consider in [1] the problem in $\mathbb{R}^3$ and present a solution that is using an embedding of the circle onto the surface of the unit ball in $\mathbb{R}^3$.

For sets of vectors that lie in *one hemisphere*, we claim — using an assumption — that the smallest enclosing cone is contained in the family of distance functions that we consider in this thesis. Additionally, we will show that the smallest enclosing cone problem can be efficiently reduced to the smallest enclosing ball problem for point sets lying in one hemisphere. In [24], Lawson claims this to be true in $\mathbb{R}^3$ though he does not provide a proof.

1.1.6 Superorthogonal Balls

In [12], Fischer introduces the concept of the smallest superorthogonal ball of balls. Smallest superorthogonal balls are related to smallest enclosing balls as they have either to contain the balls or intersect them so that *the tangent planes at every boundary intersection point span an outer angle of at least* 90 *degrees* [12]. Fischer shows that the smallest superorthogonal ball is unique and can be found efficiently. We will show that the problem of the smallest superorthogonal ball of balls is contained in the family of distance functions that we are considering here.

1.1.7 ϵ-Core Sets and Approximation

The algorithms for the computation of the smallest enclosing ball that we have mentioned so far compute the *exact* smallest enclosing ball. Instead of the exact smallest enclosing ball, one might be interested in an enclosing ball that is *not much larger* than the smallest enclosing ball. One might be willing to accept this approximation because the computation of the exact smallest enclosing ball is *too slow* for point sets in high dimensions.

For the smallest enclosing ball problem for point sets with L_2 distance functions, Bădoiu and Clarkson introduced in [4, 5] the concept of *ϵ-core sets*. They show that a subset of $\lceil \frac{1}{\epsilon} \rceil$ points is sufficient to produce an enclosing ball with a radius that is enlarged by at most $(1 + \epsilon)$ compared with the radius of the smallest enclosing ball.

The concept of ϵ-core sets has been applied to different settings. Nock and Nielsen expanded in [32] the concept of ϵ-core sets to Bregman balls. In [33], Panigrahy showed for convex polytopes of a fixed shape that can be translated and scaled that *the smallest enclosing convex polytope* of the given shape can be be approximated by an ϵ-core set. The same is shown for a convex polytope of fixed shape that can be rotated and scaled. Har-Peled and Varadarajan give in [18] an ϵ-core set algorithm for the problem of smallest k-flat radii which contains the smallest enclosing cylinder problem. In [38], Xie, Snoeyink, and Xu applied the concept of ϵ-core sets to finding the maximum inscribed sphere in a polytope.

1.2 Terminology

Distance Function For a point p, we say that the distance between p and x — *as it is perceived by* p — is denoted by $d_p(x)$ with $d_p : \mathbb{R}^d \to \mathbb{R}$.

SC Point Set Given a point set $P \subset \mathbb{R}^d$. For a point $p \in P$, let

$$L(p, t) = \{x \in \mathbb{R}^d \mid d_p(x) \leq t\}$$

be the *level set* defined by t. We say that $L(p, t)$ is strictly convex if and only if for all $x, y \in \mathbb{R}^d$ with $x \neq y$, $\lambda \in (0, 1)$, $d_p(x) \leq t$, and $d_p(y) \leq t$:

$$\begin{aligned} d_p(x) = d_p(y) &\Rightarrow d_p((1-\lambda)x + \lambda y) < (1-\lambda)d_p(x) + \lambda d_p(y) \\ d_p(x) \neq d_p(y) &\Rightarrow d_p((1-\lambda)x + \lambda y) < t. \end{aligned}$$

If $L(p,t)$ is strictly convex for all $p \in P$ and $t \in \mathbb{R}_{\geq 0}$, then we say that the point set P is a *SC point set*.

Smallest Enclosing Ball Problem For a point set $P \subset \mathbb{R}^d$, we say that the smallest enclosing ball problem is defined as finding c and r that solve the optimization problem

$$\begin{array}{ll} \text{minimize} & r \\ \text{subject to} & \forall p \in P: \ d_p(c) \leq r \\ & c \in \mathbb{R}^d. \end{array}$$

Miniball We use *miniball* as a synonym for *smallest enclosing ball*.

MB(P) For a point set P, we denote the set of miniballs by $MB(P)$.

1.3 Acknowledgement

The author is in Bernd Gärtner's debt. He guided to find the main roads, encouraged to follow tiny and windy roads, and helped find a path out of one-way streets.

The first part of this thesis follows Kaspar Fischer's work and the author is thankful for having had the possibility to use this work as a basis.

1.4 Outline

In Section 2, we define a class of distance functions that have strictly convex level sets and that can be used to build SC point sets. In Section 3, we prove four properties for SC point sets that we use in Section 4 where we show that the miniball problem for SC point sets can be solved using the MSW algorithm as it was presented in [34]. Additionally, we show that in general, the miniball problem for SC point sets cannot be solved with the help of Welzl's miniball algorithm.

By the definition of SC point sets, every point has its *own* distance function. This is new as before — except for the smallest enclosing ball of balls problem and the smallest superorthogonal ball of balls problem — only settings have been considered that use the same distance function for all points in the point set.

In Section 5, we analyze point sets with various distance functions. For each distance function, we show that it has strictly convex level sets and how to compute the smallest enclosing ball assuming that we know which points lie on the boundary of the miniball. In Section 6, we analyze two point sets that are not SC point sets and show how to compute the miniball.

In Section 7, we analyze approximate solutions for the miniball problem. First, we investigate point sets with L_2 distance functions and show that there is an *optimal* point so that removing this point produces a miniball that is shrunk by only a constant factor. We argue that this is loosely coupled with the idea of ϵ-core sets. For SC point sets in general, we show that there is no optimal point so that removing this point produces a miniball that is shrunk by only a constant factor and that there cannot be ϵ-core sets with a size that is solely dependent on ϵ. Further, we show that for SC point sets, removal of an optimal point produces a miniball with a decrease in size that is dependent on the distance functions. This has no direct application but we assume that this might be connected to the existence of core sets for SC point sets that are dependent on ϵ and the distance functions. In a last part, we present two core set algorithms and show for the first one that it is not working for SC point sets and conjecture for the second one that it is working for SC point sets.

1.5 Toy Application

In this section, we describe a toy application that can help to explain the setting considered in this thesis to *Jane and John Doe*.

Three friends are on vacation close to a lake. **A**dam loves swimming, **B**eth prefers to take a pedalo, and **C**aroline is in possession of a rowing boat. The three friends leave shore in different places and move for a while away from the shore. As being out on the lake on your own is not as much fun as being *together* out on a lake, Adam, Beth, and Caroline are looking for the spot on the lake where they can meet as soon as possible. Unfortunately, the three friends move forward at different speed. Adam is swimming a lot slower than Beth is moving forward in her pedalo, and Caroline with her rowing boat is a lot faster than Beth is.

In Figure 1, we see the positions of **A**dam, **B**eth, and **C**aroline. Additionally, we sketched with circles the distance that each of the three can move forward in 10, 20, and 26.5 minutes. As Adam is moving forward the slowest, his circles are the smallest. From Figure 1, it is immediately obvious that the friends should move towards c to meet as fast as possible. This is obvious because c is the only spot on the lake that can be reached by all three friends in at most 26.5 minutes.

It might be possible to find c with a trial and error method. But notice that for Adam $d_A(x) = \|A - x\|$, for Beth $d_B(x) = 1.3\|B - x\|$, and for Caroline $d_C(x) = 1.6\|C - x\|$ whereas only the differences in speed among

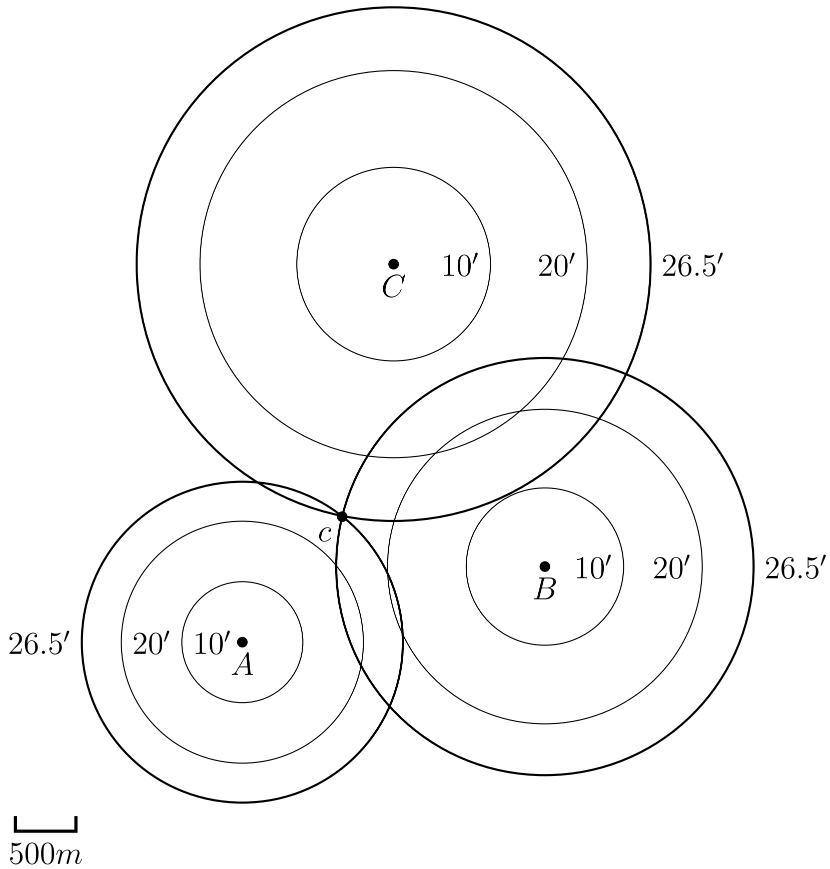


Figure 1: The position of the three friends **A**dam, **B**eth, and **C**aroline. The circles denote the distance that the friends can move forward in 10, 20, respectively 26.5 minutes. In order to meet each other as fast as possible, all friends have to move towards c.

Adam, Beth, and Caroline are important. It is easy to see that the level sets $L(A,t)$, $L(B,t)$, and $L(C,t)$ are strictly convex and therefore it is the setting that we consider in this thesis. Finding the point c can then be formulated as

$$\begin{array}{ll} \text{minimize} & r \\ \text{subject to} & \forall p \in \{A,B,C\}: \ d_p(c) \leq r \\ & c \in \mathbb{R}^2 \end{array}$$

and we can conclude that finding the spot on the lake so that the three can meet as soon as possible is a miniball problem. c is the spot where they will meet and with the help of r, the time until the meeting takes place can be determined.

Notice that we could alter the model of how the friends move forward. We could say that Adam is swimming faster in directions similar to the direction of the current and slower in directions against the current. As long as the level set of spots that can be reached by Adam in — say — 10 minutes is strictly convex, the problem of the three friends meeting is still part of the miniball problem that we consider in this thesis. Have a look at Figure 2 for a sketch of such a situation.

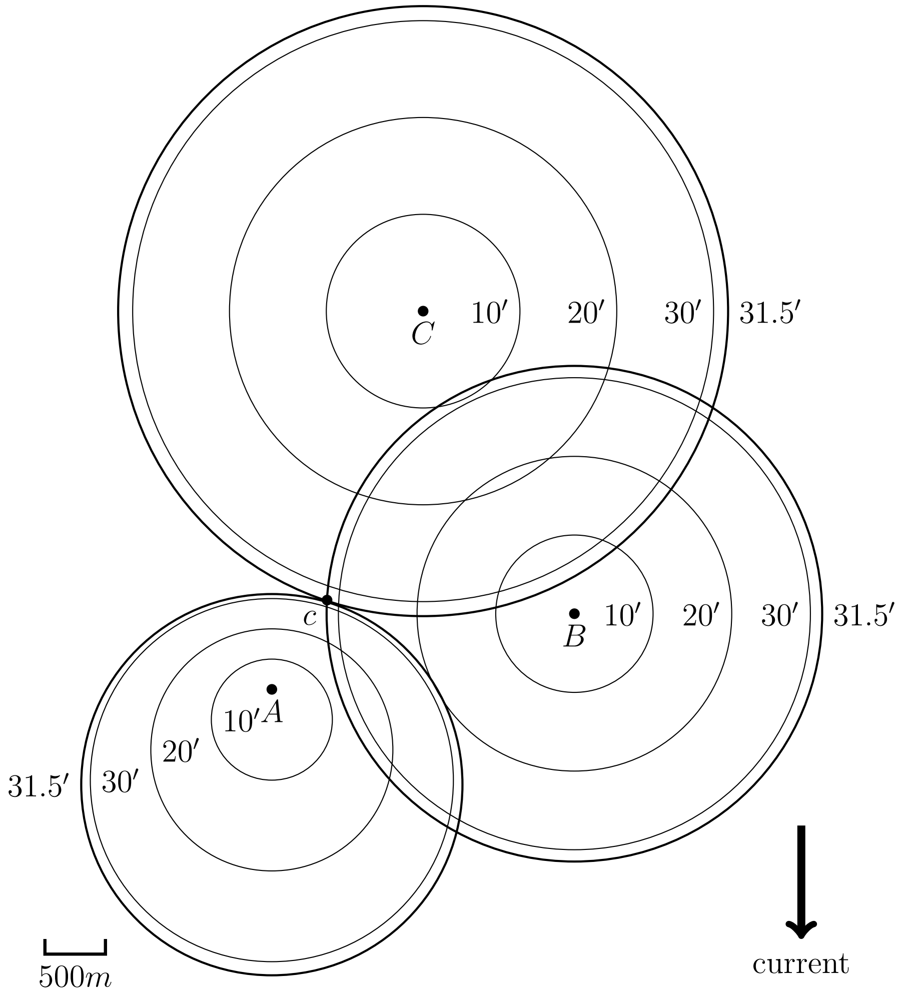


Figure 2: The three friends **A**dam, **B**eth, and **C**aroline are at the same positions as in Figure 1 though Adam is moving forward faster in directions aligned with the current and slower in opposite directions. The circles denote the distance that the friends can move forward in 10, 20, 30, respectively 31.5 minutes. To meet as fast as possible, the friends have to move towards c.

2 Setting

Let $P \subset \mathbb{R}^d$ be a finite nonempty point set with every point having its own distance function. For a point $p \in P$, we say that $d_p(c)$ is the distance between p and c *as perceived by* p.

Definition 2.1. *A point* p *is contained in a* ball *with center* c *and radius* r *if and only if* $d_p(c) \leq r$ *holds.*

Notice that the ball in Definition 2.1 does not have to be a geometrical ball. The smallest enclosing ball problem for a point set P in this setting is defined as finding c and r that solve the optimization problem

$$\begin{array}{ll} \text{minimize} & r \\ \text{subject to} & \forall p \in P: \; d_p(c) \leq r \\ & c \in \mathbb{R}^d. \end{array}$$

The properties of the miniball $MB(P)$ are dependent on the involved distance functions. We consider *SC point sets* as introduced in Section 1.2; let us repeat the definition of SC point sets. For all points $p \in P$, let

$$L(p,t) = \{x \in \mathbb{R}^d \mid d_p(x) \leq t\}$$

be the *level set* defined by t. We say that $L(p,t)$ is strictly convex if and only if for all $x, y \in \mathbb{R}^d$ with $x \neq y$, $\lambda \in (0,1)$, $d_p(x) \leq t$, and $d_p(y) \leq t$:

$$\begin{array}{lcl} d_p(x) = d_p(y) & \Rightarrow & d_p((1-\lambda)x + \lambda y) < (1-\lambda)d_p(x) + \lambda d_p(y) \\ d_p(x) \neq d_p(y) & \Rightarrow & d_p((1-\lambda)x + \lambda y) < t; \end{array}$$

consult [11] for a motivation of this definition. A point set P is an SC point set if and only if the level sets of all points are strictly convex for $t \in \mathbb{R}_{\geq 0}$. We will see that having an SC point set P allows us to show nice properties for $MB(P)$.

Obviously, strictly convex distance functions fulfilling

$$d_p((1-\lambda)x + \lambda y) < (1-\lambda)d_p(x) + \lambda d_p(y)$$

have strictly convex level sets. Let us consider point sets $P \subset \mathbb{R}^d$ so that all distance functions $d_p(c)$ are *strictly quasi-convex.* According to [8], a function d_p has to fulfill

$$\forall x, y \in \mathbb{R}^d, x \neq y, \forall \lambda \in (0,1): \; d_p((1-\lambda)x + \lambda y) < \max\{d_p(x), d_p(y)\}$$

to be strictly quasi-convex. Notice that this definition of strict quasi-convexity might be different from definitions found elsewhere; e.g. in [21].

The lemma following next proves that the level set $L(p,t)$ for strictly quasi-convex distance functions is strictly convex.

Lemma 2.2. *Given a level set $L(p,t) = \{x \in \mathbb{R}^d \,|\, d_p(x) \leq t\}$. If the distance function $d_p(x)$ is strictly quasi-convex for all $x \in \mathbb{R}^d$, then the level set $L(p,t)$ is strictly convex.*

Proof. To prove that $L(p,t)$ is strictly convex, we have to show that for two points $x \neq y$ with $d_p(x) \leq t$ and $d_p(y) \leq t$ and $\lambda \in (0,1)$,

(i) $d_p((1-\lambda)x + \lambda y) < t$ holds for x and y with $d_p(x) \neq d_p(y)$.

(ii) $d_p((1-\lambda)x + \lambda y) < (1-\lambda)d_p(x) + \lambda d_p(y)$ holds for x and y with $d_p(x) = d_p(y)$ (*strict convexity*).

Let us first show (i). We can use the strict quasi-convexity and our preconditions to get

$$\lambda \in (0,1): \; d_p((1-\lambda)x + \lambda y) < \max\{d_p(x), d_p(y)\} \leq t.$$

To prove (ii), we have to show that

$$\forall \lambda \in (0,1): \; d_p((1-\lambda)x + \lambda y) < (1-\lambda)d_p(x) + \lambda d_p(y).$$

Using the definition of strict quasi-convexity and the knowledge that both x and y have the same distance from p, we get

$$\begin{aligned} \forall \lambda \in (0,1): \; d_p((1-\lambda)x + \lambda y) &< \max\{d_p(x), d_p(y)\} \\ &= (1-\lambda)d_p(x) + \lambda d_p(y) \end{aligned}$$

and this proves the strict convexity of the level set $L(p,t)$. □

We have shown that for $t \in \mathbb{R}_{\geq 0}$, the level set $L(p,t)$ is strictly convex for the classes of *strictly convex* and *strictly quasi-convex* distance functions $d_p(x)$.

3 Properties of Miniball

In this section, we explore properties of the miniball for SC point sets.

3.1 Existence

First, we look at the existence of the miniball for SC point sets.

Lemma 3.1. *Given a finite nonempty point set P so that $L(p,t)$ is a strictly convex object for all $p \in P$ and $t \in \mathbb{R}_{\geq 0}$. Then there exists a t with*

$$t \leq \max_{p,q \in P}\{d_p(q)\} < \infty$$

so that the intersection $\bigcap_{p \in P} L(p,t)$ is nonempty. Further, there exists an enclosing ball with smallest radius.

Proof. By the definition of $d_p(c)$ as given in Section 1.2, $d_p(q) < \infty$ for all $p, q \in P$. It follows that the maximum distance t between any two points in P is smaller than ∞.

Assume that $p, q \in P$ maximize $d_p(q)$. Let $t = d_p(q)$ and therefore $L(p,t)$ contains q on its boundary and all $q' \in P$ are contained in $L(p,t)$. Notice that $P \subseteq L(p',t)$ for all $p' \in P$ as otherwise we would have chosen p and q differently. It follows that $P \subseteq C(t) = \bigcap_{p \in P} L(p,t)$ and therefore there is a nonempty intersection.

As long as the intersection $C(t)$ is nonempty, we can find a point $c \in C(t)$ so that the ball $B(c,t) = \{p \in P \mid d_p(c) \leq t\}$ with center c and radius t contains all points in P. The existence of a ball with *smallest* radius can be proved by decreasing t as long as $C(t) \neq \emptyset$. Assume that t' is the smallest value so that $C(s) \neq \emptyset$ for all $t' \leq s \leq t$ and $C(t' - \epsilon) = \emptyset$ for $\epsilon > 0$. There has to be such a t' because $L(p,t)$ is a closed set for all $p \in P$ and therefore $C(t) \neq \emptyset$ is as well a closed set. By the convexity of $L(p,t)$ and the fact that for $s' < s$, $L(p,s') \subseteq L(p,s)$, we can conclude that there cannot be an $s < t'$ with $C(s) \neq \emptyset$. □

3.2 Uniqueness

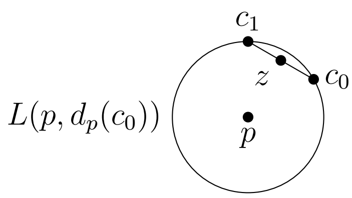
With Lemma 3.1, we know that there exists a smallest enclosing ball $MB(P)$ with center c and radius r. Assume that we are given such a miniball. As we have already seen in the proof of Lemma 3.1, for every point $p \in P$, the center c has to be contained in the level set $L(p,r)$; equivalently, $c \in \bigcap_{p \in P} L(p,r)$. We use this to prove uniqueness.

Lemma 3.2. *Given a finite nonempty set of points P so that $L(p,t)$ is a strictly convex object for all $p \in P$ and $t \in \mathbb{R}_{\geq 0}$. Then there is a unique smallest enclosing ball for the point set P.*

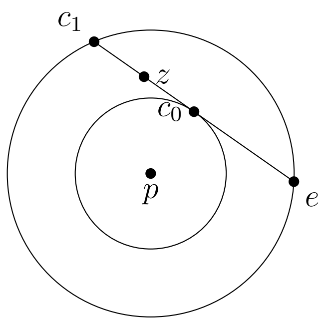
Proof. With Lemma 3.1, we know that there is a smallest enclosing ball with radius r. If $r = 0$, uniqueness is trivial and we assume that $r \neq 0$. Further assume that there are at least two smallest enclosing balls with radius r and centers $c_0 \neq c_1$.

We know that $d_p(c_0) \leq r$ and $d_p(c_1) \leq r$ has to hold for all $p \in P$. Let us distinguish two cases dependent on p:

i. If $d_p(c_0) = d_p(c_1)$, we know by strict convexity of $L(p,r)$ that for all $\lambda \in (0,1)$, $z = (1-\lambda)c_0 + \lambda c_1$ and $d_p(z) < d_p(c_0)$ holds. With $d_p(c_0) \leq r$, we can conclude that $d_p(z) < r$.



ii. If $d_p(c_0) \neq d_p(c_1)$, we assume w.l.o.g. that $d_p(c_0) < d_p(c_1)$. With $d_p(c_1) \leq r$, $d_p(c_0) < d_p(c_1) \leq r$ follows. $L(p,t)$ is a strictly convex object for all $t \in \mathbb{R}_{\geq 0}$ and it follows that c_0 lies inside $L(p, d_p(c_1))$. Further, we can find the point $e \neq c_1$ that lies on the line defined by c_0 and c_1 and that has $d_p(e) = d_p(c_1)$. Due to the strict convexity of $L(p, d_p(c_1))$ we know that for $z = (1-\lambda)c_0 + \lambda c_1 = (1-\mu)e + \mu c_1$ for $\mu \in (0,1)$, we have $d_p(z) < d_p(c_1) \leq r$.



It follows that we can find a point z so that for every $p \in P$, $d_p(z) < r$. This is a contradiction to our assumption that the smallest enclosing ball has radius r and it follows that the smallest enclosing ball is unique. □

3.3 Combinatorial Dimension

In the context of combinatorial dimension, we prove two properties. The first property states that if a point $p \in P$ is properly contained in the smallest enclosing ball, then

$$MB(P) = MB(P \setminus \{p\}).$$

The second property gives us an upper bound on the number of points that define the smallest enclosing ball and this property is called combinatorial dimension.

Lemma 3.3. *If a point $p \in P$ is properly contained in the smallest enclosing ball $MB(P)$, then $MB(P) = MB(P \setminus \{p\})$.*

Proof. For a point $p \in P$ that is properly contained in $MB(P)$ with center c and radius r, $d_p(c) < r$ holds by definition. It follows that c is properly contained in $L(p, r)$. By the uniqueness of $MB(P)$, we know that $C(r) = \bigcap_{p \in P} L(p, r)$ contains only c and we can conclude that $C(r)$ is properly contained $L(p, r)$.

Because $C(r)$ is properly contained in $L(p, r)$, it is not possible that $L(p, r)$ defines part of the boundary of $C(r)$. Further, the intersection of strictly convex objects is again a strictly convex object; e.g. it is *not* possible that two disconnected objects are contained in $C(r)$. Notice that for level sets that are *not* strictly convex, it is possible that $C(r)$ contains multiple *disconnected* regions and removing the point p properly contained in $MB(P)$ might alter $C(r)$. It follows that removing the point p does not change the intersection $C'(r) = \bigcap_{p' \in P \setminus \{p\}} L(p', r)$; $C(r) = C'(r)$ holds and the center of the miniball has to be the same after removal of p. Because $d_p(c) < r$, there has to be a $q \neq p$ with $d_q(c) = r$ and we can conclude that the radius remains as well the same. □

Lemma 3.4. *Let $P \subset \mathbb{R}^d$ be a set of at least $d + 1$ points. There exists a subset $P' \subseteq P$ with $|P'| = d + 1$ points fulfilling $MB(P') = MB(P)$.*

Proof. Given a point set P with smallest enclosing ball $MB(P)$. The smallest enclosing ball has center c and radius r. We know that

$$\{c\} = C(r) = \bigcap_{p \in P} L(p, r)$$

by Lemmata 3.1 and 3.2. It follows that

$$C'(r) = \bigcap_{p \in P} \operatorname{int}(L(p, r)) = \bigcap_{p \in P} \{x \in \mathbb{R}^d \mid d_p(x) < t\} = \emptyset.$$

To conclude our proof, we make use of Helly's theorem [9]. Given convex sets $S_i \subset \mathbb{R}^d$ with $i \in [m]$ and $m > d+1$. If for all possible $J \subset [m]$ with $|J| = d+1$, $\bigcap_{j \in J} S_j \neq \emptyset$ holds, then $\bigcap_{i \in [m]} S_i \neq \emptyset$ follows.

As $\bigcap_{p \in P} \operatorname{int}(L(p,r)) = \emptyset$, we can conclude with Helly's theorem that there is a set P' with $|P'| = d+1$ so that $\bigcap_{p \in P'} \operatorname{int}(L(p,r)) = \emptyset$. It follows that it is not possible that there is a ball with radius $r' < r$ that encloses all points in P' and combining this with $P' \subset P$, we know that $MB(P)$ and $MB(P')$ have to have the same radius. By uniqueness for the set P', we can argue that $MB(P') = MB(P)$. □

Lemmata 3.3 and 3.4 push us towards the concept of a *basis*; let us introduce this concept formally.

Definition 3.5. *Given a point set P, we call $V \subseteq P$ a basis of P if and only if $MB(V) = MB(P)$ and there is no point set $V' \subseteq P$ with $MB(V') = MB(P)$ and $|V'| < |V|$. Notice that it is possible to have more than one basis and that the size of the basis can be smaller than $d+1$.*

3.4 Miniball Problem is an LP-Type Problem

The generic algorithm presented in the next section will make use of the fact that the smallest enclosing ball problem for SC point sets is an *LP-type problem* as it was introduced by Sharir and Welzl in [34].

Definition 3.6. ***(Adaption from [34])*** *Let H be a set of constraints and ω be a function mapping every subset G of H to its optimal solution with*

$$\omega : 2^H \rightarrow \Omega$$

whereas Ω is a set with linear order $<$. We call (H, ω) an LP-type problem if the two conditions

1. *$F \subseteq G \subseteq H$ implies $\omega(F) \leq \omega(G)$.*

2. *$F \subseteq G \subseteq H, \omega(F) = \omega(G), h \in H$ implies $\omega(F+h) > \omega(F) \Leftrightarrow \omega(G+h) > \omega(G)$*

are fulfilled.

Let us show that the miniball problem for SC point sets is indeed an LP-type problem.

Lemma 3.7. *Let $P \subset \mathbb{R}^d$ be a finite nonempty SC point set and define $w : 2^P \rightarrow \mathbb{R}$ to be the radius of the smallest enclosing ball; e.g. for $N \subseteq P$, $\omega(N)$ is the radius of $MB(N)$. (P, ω) is an LP-type problem with maximum combinatorial dimension $d+1$.*

Proof. As $\omega(N)$ is the radius of $MB(N)$ for an $N \subseteq P$, the set Ω as presented in Definition 3.6 has linear order and we are left to show that the two conditions from Definition 3.6 hold.

For the first condition, we assume that for $N \subseteq O \subseteq P$, $\omega(N) \leq \omega(O)$ does *not* hold and show a contradiction. As $\omega(N)$ is the radius of $MB(N)$ respectively $\omega(O)$ the radius of $MB(O)$, it follows that $MB(N)$ has to have a larger radius than $MB(O)$. This is not possible because N is contained in O and therefore as well in $MB(O)$; a contradiction.

For the second condition, we observe that if $N = O$, then locality is trivial and we assume that $N \subset O$. With $\omega(N) = \omega(O)$, $N \subset O$, and uniqueness of the miniball for N, we know that $MB(N)$ and $MB(O)$ are the same miniballs. With $MB(N)$ and $MB(O)$ being the same miniballs, locality follows trivially.

With Lemma 3.4, an upper bound of $d+1$ on the combinatorial dimension follows immediately. □

In the next section, we use the fact that the miniball problem for SC point sets is an LP-type problem.

4 Algorithms

With the help of the properties that we have shown in the last section, we present an algorithm that solves the miniball problem for SC point sets.

First, we present Welzl's algorithm for the miniball problem with points in the Euclidean setting [37] and show why this algorithm cannot be applied to our setting. In a second part, we show that the miniball problem for SC point sets can be solved with the MSW algorithm as described in [34].

4.1 Welzl's Miniball Algorithm

Given a point set P with $p \in P$ having the distance function

$$d_p(x) = \|p - x\|_2,$$

Welzl presents in [37] an algorithm to solve the smallest enclosing ball problem. He uses a lemma to show the correctness of the algorithm; we list here the first part of this lemma.

Lemma 4.1. ***(Adaption of Lemma 1 from [37])*** *Let P and R be finite point sets in the plane and P nonempty. Assume that b_md(P, R) is the ball of smallest radius that contains all points in P and that has all points in R on the boundary. If there exists a disk containing P with R on its boundary, then b_md(P, R) is well-defined (unique).*

Based on Lemma 4.1, Welzl shows that Algorithm 1 solves the smallest enclosing ball problem.

```
procedure seb(U, V)
begin
    if U = V then
        return any ball from the set MB(V, V)
    else
        choose B ∈ U \ V uniformly at random
        D = seb(U \ {B}, V)
        if B ⊈ D then
            return seb(U, V ∪ {B})
        else
            return D
        end
    end
end
```

Algorithm 1: Welzl's miniball algorithm

We show that there is an SC point set that does not fulfill the uniqueness property as given in Lemma 4.1. Given a set of points P with

$$d_p(x) = s_p \|p - x\|_2$$

for all $p \in P$ and for a scale factor $s_p \in \mathbb{R}_{\geq 0}$. Every level set $L(p,t)$ is clearly a ball and strict convexity follows[2].

Assume that $R = P \subset \mathbb{R}^2$ and $|P| = 3$. Further assume that the points in P lie on a common line l and that the scale factors s_p are not yet defined. Let us choose a center c_0 strictly on one side of the line l. Because the points lie on the boundary of the miniball, we know that

$$\forall p \in P: \; d_p(c) = r$$

for a radius r. We choose $r > 0$ arbitrarily and with this choice, the scale factors s_p for all $p \in P$ are determined. But now, due to symmetry, we can find a miniball with center $c_1 \neq c_0$ by reflecting c_0 at the line l so that all points in P are on the boundary of this new miniball.

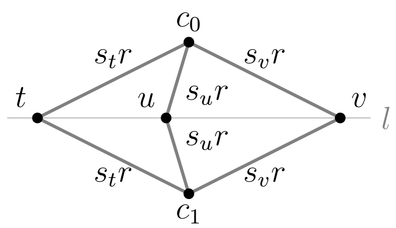


Figure 3: Example of a point set $P = \{t, u, v\}$ that is lying on the boundary of *two* smallest enclosing balls with centers $c_0 \neq c_1$.

We have shown that the miniball for a set P having a set $R \subseteq P$ on its boundary is not unique though this does not directly imply that Algorithm 1 is not working.

In [12], Fischer presents an example for the smallest enclosing ball of balls problem showing that Welzl's algorithm is not working. The smallest enclosing ball of balls problem is contained in the smallest enclosing ball of SC point set problem[3] and it follows that in general, Welzl's algorithm cannot be applied to SC point sets.

[2] a proof is provided in Section 5.2

[3] a proof is provided in Section 5.4

4.2 MSW

In [34], Sharir and Welzl present a generic algorithm that solves LP-type problems given the implementation of two primitive operations. In Lemma 3.7, we have shown that the miniball problem for SC point sets is an LP-type problem. The MSW algorithm introduced in [27] is given as Algorithm 2.

procedure $msw(U, V)$
`// Computes a basis of` U
`// Precondition:` $V \subseteq U$ `is a basis`
begin
    if $U = V$ **then**
        return V
    else
        choose $x \in U \setminus V$ uniformly at random
        $W = msw(U \setminus \{x\}, V)$
        if $violates(x, W)$ **then**
            return $msw(U, basis(W, x))$
        else
            return W
        end
    end
end

Algorithm 2: MSW to solve LP-type problems

The MSW algorithm (Algorithm 2) and the miniball algorithm (Algorithm 1) have a pretty similar structure. The difference between the two algorithms is that the MSW algorithm uses operations *basis* and *violates* whereas the miniball algorithm uses properties of the physical ball directly. The two operations are defined as follows:

- *violates:* For a basis $V \subseteq U \subseteq P$ and a constraint $x \in U \setminus V$, $violates(x, V)$ returns *yes* if and only if x makes it impossible for V to be a basis of $V \cup \{x\}$.
- *basis:* For a basis $V \subseteq U \subseteq P$ and a constraint $x \in U \setminus V$, $basis(V, x)$ returns a basis of the point set $V \cup \{x\}$.

Let us give possible implementations for the two operations.

Violates The implementation of *violates* depends on the distance functions $d_p(c)$ for the points $p \in P$. To test whether a point lies inside or outside a miniball, we have at least two possibilities:

1. For a basis V with $|V| \leq d+1$, compute the miniball. The center c and the radius r are determined by the system of equations

$$\forall p \in V : d_p(c) = r. \tag{1}$$

   This system of equations is correct because all points of a basis lie on the boundary of the miniball. If V is a basis, then there has to be a miniball and this miniball has to be unique. It follows that the system of equations has a solution $c \in \mathbb{R}^d$ that minimizes $r \in \mathbb{R}_{\geq 0}$ and that this solution is unique. The difficulty of solving the System of Equations 1 depends on the distance functions $d_p(c)$. Given c, r, and the d_p's, it is easy to test whether a point q lies inside ($d_q(c) \leq r$) or outside ($d_q(c) > r$) the miniball. Notice that the test whether a point lies inside or outside the ball might not be obvious to do in rational arithmetic.

2. Use geometric properties of the miniball and its basis. This approach is taken in [16] where the test for containment in an ellipse is carried out in rational arithmetic even though the center point of the ellipse can be irrational.

Basis Assuming that there is an implementation for *violates*, Fischer shows in [12] that *basis* can be computed:

Lemma 3.3 states that if a point x is properly contained in $MB(V)$, then $MB(V) = MB(V \setminus \{x\})$. In the basis computation, we are given a basis W and a point x that is not contained in the miniball defined by this basis; this can be formulated as $MB(W \cup \{x\}) \neq MB(W)$. With Lemma 3.3, it follows that x lies on the boundary of the miniball $MB(W \cup \{x\})$. We can employ this property and construct the basis for a point set $W \cup \{x\}$ in a brute-force manner:

Recalling Definition 3.5, a basis of P has to be a set of points $V \subseteq P$ of *minimum size* so that $MB(V) = MB(P)$. Using the property of minimum size, we can generate all point sets $V' \subseteq W$ according to their size. Starting with a set V' with smallest size, we test whether $V = V' \cup \{x\}$ is a basis of $W \cup \{x\}$; if we are given an implementation of the *violates* operation, this test is easy. The first point set V that has $MB(V) = MB(W \cup \{x\})$ is a basis because we generate the point sets V in increasing size.

It is important to notice that this implementation is brute-force and it might be possible to find better implementations for specific instances of the problem. In [12], Fischer does so for the case of the smallest enclosing ball of balls.

5 SC Point Sets and the Miniball for a Given Basis

After having presented the properties of SC point sets and an algorithm that allows us to compute the miniball for SC point sets, we present specific instances of distance functions. We show that the distance functions have strictly convex level sets $L(p, t)$ and elaborate on how to compute the miniball for a given basis of the point set. With the help of the miniball, the implementation of the *violates* operation follows immediately.

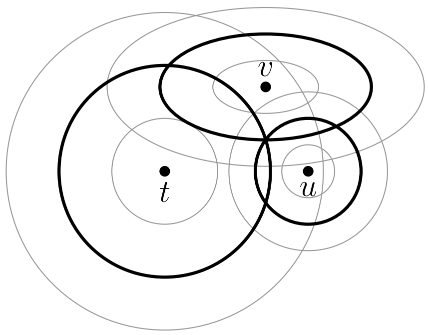


Figure 4: Miniball for points t, u, and v having distance functions $d_t(x) = \frac{\|t-x\|}{2}$, $d_u(x) = \|u - x\|$, and $d_v(x) = \sqrt{\frac{|v_0-x_0|^2}{2} + \frac{|v_1-x_1|^2}{1}}$. The thick lines illustrate the intersection of the level sets defining the miniball.

Notice that in the work presented so far, we have not been assuming that the points in P use the same distance functions. As long as all points in P use a distance function with strictly convex level sets, we can apply algorithm MSW — based on the condition that we can implement the *violates* operation. Further, we have not made any conditions on the *distribution* of the distance functions among the points; it is not required that nearby points in P use similar distance functions. For an illustration of a miniball for a point set with different distance functions, have a look at Figure 4.

5.1 Points with L_p $(1 < p < \infty)$ Distance Functions

For $1 < p < \infty$, we consider points $q \in P$ with distance functions

$$d_q(c) = \|c - q\|_p := \sqrt[p]{\sum |c_i - q_i|^p}.$$

We show that $\|c - q\|_p$ is a distance measure with strictly convex level sets and then hint at how to implement the *violates* operation.

Strict Convexity of Level Sets

We show strict convexity of the level sets for L_p distance functions with $1 < p < \infty$.

Lemma 5.1. *The distance function $d_q(x) = \|x - q\|_p$ with $1 < p < \infty$ has strictly convex level sets.*

Proof. We show that the distance function $d_q(x) = \|x - q\|_p$ is a strictly quasi-convex distance function. With Lemma 2.2, this implies that the level set is strictly convex. A distance function $d_q(x)$ is strictly quasi-convex if

$$\forall x, y \in \mathbb{R}^d, x \neq y, \forall \lambda \in (0,1) : \; d_q((1-\lambda)x + \lambda y) < \max\{d_q(x), d_q(y)\}$$

holds. From the definition of $d_q(x)$, it follows that $d_q(x) = d_{q-y}(x - y)$ and we can therefore translate our point set so that $q = \mathbf{0}$. By definition of $d_0(x) = \sqrt[p]{\sum |x_i|^p}$, we can conclude that $ad_0(x) = d_0(ax)$ and therefore assume that

$$x, y \in U = \{z \in \mathbb{R}^d \mid \|z\|_p \leq 1\}.$$

The condition for strict quasi-convexity can then be reformulated as

$$\forall \lambda \in (0,1), \; x, y \in U : \; \|(1-\lambda)x + \lambda y\|_p < \max\{\|x\|_p, \|y\|_p\}.$$

Assume that $x, y \in U$, $x \neq y$, and as we can scale, we can assume that $\|x\|_p = 1$ and $\|y\|_p \leq 1$. We remind ourselves of the definition of the L_p norm

$$\|(1-\lambda)x + \lambda y\|_p = \sqrt[p]{\sum |(1-\lambda)x_i + \lambda y_i|^p}.$$

If we could show that for at least one i

$$|(1-\lambda)x_i + \lambda y_i|^p < (1-\lambda)|x_i|^p + \lambda |y_i|^p$$

and that for all other i's

$$|(1-\lambda)x_i + \lambda y_i|^p \leq (1-\lambda)|x_i|^p + \lambda |y_i|^p$$

holds, then we could conclude that

$$\begin{aligned}
\|(1-\lambda)x + \lambda y\|_p &= \sqrt[p]{\sum |(1-\lambda)x_i + \lambda y_i|^p} \\
&< \sqrt[p]{(1-\lambda)\sum |x_i|^p + \lambda \sum |y_i|^p} \\
&= \sqrt[p]{(1-\lambda)\|x\|_p^p + \lambda \|y\|_p^p} \\
&\leq 1 \\
&= \|x\|_p \\
&= \max\{\|x\|_p, \|y\|_p\}
\end{aligned}$$

and we had shown that $d_q(x)$ is a strictly quasi-convex distance function and that therefore the level sets of $d_q(x)$ are strictly convex.

All there is to do is to prove the strict convexity of $|x|^p$. To do so, divide the problems into subproblems that are dependent on the values of x and y.

First assume that $x \neq y$, $x > 0$, and $y > 0$. For any point

$$\lambda \in (0, 1) : \; l(\lambda) = (1 - \lambda)x + \lambda y,$$

$z = l(\lambda)$ is strictly positive. So for a strictly positive z, $|z|^p = z^p$ and differentiating this twice yields $(p-1)pz^{p-2}$. Because we were assuming that $p > 1$, $(p-1)pz^{p-2}$ is strictly positive for $z \in (0, \infty)$ and we can therefore conclude that the function $|z|^p$ is strictly convex for $z \in (0, \infty)$ [36]. It follows that $|(1-\lambda)x + \lambda y|^p < (1-\lambda)|x|^p + \lambda|y|^p$ holds for $x, y \in \mathbb{R}_{>0}$.

For $x \neq y$, $x < 0$, and $y < 0$ we can argue in the same way. $|z|^p = (-z)^p$ and the second derivative is $(p-1)p(-z)^{p-2}$ which is again strictly positive assuming $p > 1$. We can conclude that strict convexity $|(1-\lambda)x + \lambda y|^p < (1-\lambda)|x|^p + \lambda|y|^p$ holds for $x, y \in \mathbb{R}_{<0}$ [36].

Next, let us look at the situation with $x = 0$ and $y \neq 0$. We have

$$|(1-\lambda)x + \lambda y|^p = |\lambda y|^p = \lambda^p |y|^p.$$

By our definition $p > 1$, $\lambda \in (0, 1)$ whereas it follows that

$$\lambda^p |y|^p < \lambda |y|^p$$

holds. We can conclude that strict convexity holds as well in this case.

In a last step, let us have a look at the case with $x < 0$ and $y > 0$. Notice that this case cannot be directly reduced to the previous ones as the function $|z|^p$ is not differentiable in the interval $(-\infty, \infty)$. If λ so that $(1-\lambda)x + \lambda y = 0$, then $|(1-\lambda)x + \lambda y|^p = |0|^p$ and this is trivially strictly smaller than $(1-\lambda)|x|^p + \lambda|y|^p$; strict convexity is shown for this case.

If λ so that $(1-\lambda)x + \lambda y \neq 0$, then we first assume that $(1-\lambda)x + \lambda y > 0$. Further assume that $a = (1-\lambda)x + \lambda y$ and therefore for a μ with $a = (1-\mu)0 + \mu y$, we have

$$\mu - \lambda = \frac{a}{y} - \frac{a-x}{y-x} = \frac{x(y-a)}{y(y-x)}.$$

By our choices, we know that $y - a > 0$, $y > 0$, $y - x > 0$, and $x < 0$. It follows that our expression is strictly negative and we can conclude that $\mu - \lambda < 0$ or equivalently $\mu < \lambda$. With this, we can conclude that

$$|(1-\lambda)x + \lambda y|^p = |\mu y|^p < \mu |y|^p < \lambda |y|^p \leq (1-\lambda)|x|^p + \lambda |y|^p$$

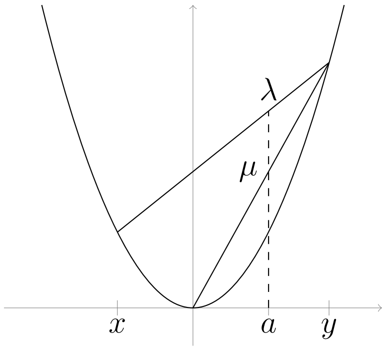


Figure 5: Strict convexity for $x < 0$, $y > 0$, and $a > 0$. λ and μ denote the points $(a, (1-\lambda)|x|^p + \lambda|y|^p)$ resp. $(a, \lambda|y|^p)$.

and strict convexity is as well proven for this case. $(1-\lambda)x + \lambda y < 0$ is handled analogously and we can conclude that for all combinations of $x \neq y$, and $\lambda \in (0, 1)$ we have

$$|(1-\lambda)x + \lambda y|^p < (1-\lambda)|x|^p + \lambda|y|^p$$

and strict convexity holds.

Obviously, strict convexity holds only for $x \neq y$. By our assumption that $x' \neq y'$ for $x', y' \in \mathbb{R}^d$, we know that there are $x \neq y$ and it follows that strict convexity holds for at least one x and y. For $x = y$,

$$|(1-\lambda)x + \lambda y|^p = |x|^p = (1-\lambda)|x|^p + \lambda|y|^p$$

and this concludes our proof for the strict convexity of the level sets of $d_q(x) = \|x - q\|_p$. □

Violates Operation

The implementation of the *violates* operation is dependent on the distance function L_p. L_2 is the traditional miniball problem as described in [37] and we do not have to bother about implementation of the *violates* operation. For all other instances of L_p, computing the center and the radius of a *small* basis boils down to solving a system of equations. Solving the system of equations can be done using methods such as [25] though this might be expensive.

5.2 Points with Scaled L_2 Distance Functions

In the instance considered so far, all points had the same distance function. In the setting considered here, the distance functions of the points have the same form but differ from each other in a parameter that is dependent on the point; the distance function

$$d_p(c) = \frac{\|p - c\|_2}{s_p}$$

is dependent on a scale factor $s_p > 0$. The property of strict convexity is invariant under scaling and we can conclude that the scaled version of L_2 fits into the framework of SC point sets.

Violates Operation

To show how to implement the violates operation, we have to prove a few properties that hold for the instance at hand but do not hold in the general case. We follow closely Fischer's strategy for the smallest enclosing ball of balls problem.

We prove the first lemma for a general type of distance functions $d_p(c)$ that contains the scaled version considered here. While proving this lemma, we will see that we can use the exact same proof strategy as in [12]. Notice that we were not able to find a reduction of the setting of scaled L_2 distance functions to the setting of finding the smallest enclosing ball of balls.

Lemma 5.2. ***(Adaption of Lemma 3.3 from [12])*** *Assume that $f_p(x)$ is a strictly increasing function; $x, y \in \mathbb{R}_{\geq 0} : x < y \Rightarrow f_p(x) < f_p(y)$. Let V be a nonempty set of points with distance function $d_p(c) = f_p(\|p - c\|_2)$ — meaning that all level sets $L(p, t)$ are balls — and D a ball that contains all points in V on its boundary. Then $D = MB(V)$ if and only if the center c_D of D is contained in the convex hull of V.*

Proof. Let us first tackle direction ($\Leftarrow$) by assuming that $D \neq MB(V)$. Say that $MB(V)$ has center $c_{D'}$ and radius $r_{D'}$. By uniqueness of $MB(V)$, this implies that $r_{D'} < r_D$ and by the assumption that all points lie on the boundary of D, $c_{D'} \neq c_D$ follows. Let us write $c_{D'} = c_D + \lambda u$ for $\lambda > 0$ and for an arbitrary unit vector u. By δ_p, let us denote the distance between $c_{D'}$ and $p \in V$. We can write

$$\delta_p = f_p\left(\|c_{D'} - p\|_2\right) = f_p\left(\sqrt{\|c_D - p\|_2^2 + \lambda^2 - 2\lambda(p - c_D)^T u}\right)$$

and because all points $p \in V$ have to be contained in the enclosing ball D', we know that

$$r_{D'} \geq \max_{p \in V} \delta_p.$$

Seidel's observation states that *a point* $q \in \mathbb{R}^d$ *lies in the convex hull* $conv(V)$ *of a finite point set* $V \subset \mathbb{R}^d$ *if and only if* $\min_{p \in V}(p-q)^T u \leq 0$ *for all unit vectors* u [12]. From this observation, we know that for the u that we have chosen to define $c_{D'} = c_D + \lambda u$, there is a p' so that $(p' - c_D)^T u \leq 0$. For this p', we have

$$\delta_{p'} = f_{p'}\left(\sqrt{\|c_D - p'\|^2 + \lambda^2 - 2\lambda(p' - c_D)^T u}\right)$$

with

$$\lambda^2 - 2\lambda(p' - c_D)^T u > 0.$$

We know that $r_D = f_{p'}(\|c_D - p'\|_2)$ and can therefore conclude that $\delta_{p'} > r_D$. Putting the bits and pieces together we get

$$r_{D'} \geq \max_{p \in V} \delta_p \geq \delta_{p'} > r_D$$

contradicting our initial assumption $r_{D'} < r_D$.

For direction ($\Rightarrow$), we assume that c_D is not contained in $conv(V)$. By Seidel's observation, we know that there is a u so that

$$\forall p \in V : (p - c_D)^T u > 0.$$

With the help of that u, construct a $c_{D'} = c_D + \lambda u$ with

$$0 < \lambda < 2 \min_{p \in V}(p - c_D)^T u.$$

We can rewrite

$$\delta_p = f_p\left(\sqrt{\|c_D - p\|^2 + \lambda(\lambda - 2(p - c_D)^T u)}\right)$$

and know that

$$\forall p \in V : \lambda(\lambda - 2(p - c_D)^T u) < 0$$

whereas we can follow that

$$\forall p \in V : \delta_p < r_D.$$

We can conclude that there is a scaled disk with center $c_{D'}$ and radius $r_{D'} = \max_{p \in V} \delta_p < r_D$ enclosing all points $p \in V$; a contradiction to the assumption that $D = MB(V)$. □

It is trivial to see that the function $f_p(x) = \frac{x}{s_p}$ is a strictly increasing function and therefore Lemma 5.2 can be applied to the scaled setting. Notice that the functions $t \in \mathbb{R}_{\geq 1} : f_p(x) = x^t$, $f_p(x) = \frac{x}{s_p} + r_p$, and $t \in \mathbb{R}_{>1} : f_p(x) = t^x$ are strictly increasing and therefore Lemma 5.2 applies as well to them.

We use the fact that the center of the miniball $MB(V)$ has to be contained in the convex sets of the basis V to prove that the points $p \in V$ are affinely independent. The proof of the next lemma is equivalent to the one given in [12], we restate it here for convenience.

Lemma 5.3. ***(Adaption of Lemma 3.8 from [12])*** *For a point set P with points having distance functions $d_p(c) = f_p(\|p - c\|_2)$ for a strictly increasing function f_p, the centers of a basis $V \subseteq P$ are affinely independent.*

Proof. In [12], Fischer uses the fact that in his setting, the center of the miniball lies in the convex hull of the basis points; by Lemma 5.2, the exact same is true for our setting.

If $P = \emptyset$, then the claim is trivially true and we assume that $P \neq \emptyset$ and therefore as well $V \neq \emptyset$ and with Lemma 5.2, the center c of a miniball $MB(V)$ can be written as $c = \sum \lambda_i p_i$ for $V = \{p_1, ..., p_m\}$. By definition, a basis is minimal and we can conclude that $\forall i : \lambda_i > 0$.

Assume that the centers are *affinely dependent*, meaning that there are μ_i so that

$$\sum \mu_i p_i = 0, \quad \sum \mu_i = 0.$$

Using this, we rewrite c as

$$c = \sum \lambda_i p_i + \alpha \sum \mu_i p_i = \sum (\lambda_i + \alpha \mu_i) p_i.$$

Set $\alpha = 0$ and increase it until for the first time the equality $\lambda_i + \alpha\mu_i = 0$ is fulfilled for some i. Because p_i is the first point that fulfills that condition, it follows that all other $p \in V \setminus \{p_i\}$ have a factor $\lambda_i + \alpha\mu_i \geq 0$. By construction, $\sum(\lambda_i + \alpha\mu_i) = 1$; a contradiction to the assumption that V is a basis and therefore minimal. □

We use this proof to show that there is a fast and exact way of computing $MB(V)$ for a basis V of P.

Lemma 5.4. ***(Adaption of Lemma 5.2 from [12])*** *Let $V \subseteq P$ be sets of points with distance function $d_p(c) = \frac{\|p-c\|_2}{s_p}$ and assume that V is a basis of P. With $MB(V, V)$, let us denote the set of miniballs of the point set V that contains V on its boundary. Then $MB(V, V) = MB(P)$ and $MB(P)$ can be computed in time $0(d^3)$.*

Proof. We have to show two things:

(i) $MB(V, V) = MB(P)$ for a basis $V \subseteq P$ of a point set P

(ii) $MB(P)$ can be computed in time $O(d^3)$.

$V = \emptyset$ implies that $P = \emptyset$ and we can conclude that $MB(\emptyset)$ is the empty ball $-\bowtie$. $MB(\emptyset, \emptyset) = MB(\emptyset)$ holds trivially and the computation of the empty ball $-\bowtie$ is clearly possible in $O(d^3)$.

Assume that $V \neq \emptyset$ and let us first show (i). V is a basis of P and $MB(V) = MB(P)$ has to hold. With Lemma 3.3, all $p \in V$ lie on the boundary of $MB(P)$ and $MB(V) \in MB(V, V)$. All balls in $MB(V, V)$ are smallest enclosing balls and have the same radius. With Lemma 3.2, we know that this smallest enclosing ball is unique and therefore $MB(V) = MB(V, V)$.

To show (ii), we assume that $V = \{p_1, ..., p_m\}$ for $m \leq d + 1$. The ball $MB(V, V)$ has center c and radius r if and only if $r \geq 0$, $\|c - p_i\|_2^2 = r^2 s_{p_i}^2$ for all i, and we have chosen the smallest r fulfilling these conditions. Define $z = c - p_1$ and for $1 < i \leq m$, $z_{p_i} = p_i - p_1$. We can reformulate the conditions for c and r as

$$\begin{aligned} z^T z &= r^2 s_{p_1}^2 \\ (z_{p_i} - z)^T (z_{p_i} - z) &= r^2 s_{p_i}^2, \ i \in [m] \setminus \{1\}. \end{aligned} \tag{2}$$

Subtracting the former from the later gives us $m - 1$ *linear* equations in z of the form

$$z_{p_i}^T z_{p_i} - 2z^T z_{p_i} = r^2 (s_{p_i}^2 - s_{p_1}^2), \ i \in [m] \setminus \{1\}. \tag{3}$$

Lemma 5.2 states that for the miniball $MB(V, V)$ with center c and radius r, $c = \sum \lambda_i p_i$ with $\sum \lambda_i = 1$ holds for some λ_i. This can be used to write

$$z = c - p_1 = \sum_{i=1}^{m} \lambda_i p_i - p_1 = \sum_{i=2}^{m} \lambda_i p_i - \sum_{i=2}^{m} \lambda_i p_1 = \sum_{i=2}^{m} \lambda_i (p_i - p_1) = Q\lambda$$

with $Q = (z_{p_2}, ..., z_{p_m})$ and $\lambda = (\lambda_2, ..., \lambda_m)$. Using this notation, we rewrite Equation 3 as

$$2z_{p_i}^T Q\lambda = z_{p_i}^T z_{p_i} + r^2 (s_{p_1}^2 - s_{p_i}^2), \ i \in [m] \setminus \{1\} \tag{4}$$

and conclude that this is a linear system of the form $A\lambda = e + r^2 f$ with $A = 2Q^T Q$.

With Lemma 5.3, we know that the points p_i for $i \in [m]$ are affinely independent and this implies that $z_{p_i} = p_i - p_1$ are linearly independent. Assume not and that z_{p_i} are not linearly independent. Then this implies that there are a_i so that

$$\sum_{i=2}^{m} a_i z_{p_i} = \sum_{i=2}^{m} a_i p_i - \sum_{i=2}^{m} a_i p_1 = 0$$

and $\sum a_i - \sum a_i = 0$ lets us conclude that the points p_i are affinely dependent; a contradiction and we can conclude that the columns of Q are linearly independent.

$Q^T Q$ is regular if the columns of Q are linearly independent. This holds because if $Q^T Qx = 0$ with $x \neq 0$, then $0 = xQ^T Qx = \|Qx\|^2$ and therefore $Qx = 0$ contradicting the linear independence of the columns of Q.

The solution space for this linear system is parameterized by r^2 and can be found in $O(d^3)$ by a standard method. Possible solutions have to fulfill $r > 0$, $\lambda_i \geq 0$, and $\lambda_1 = 1 - \sum_{i=2}^{m} \lambda_i$. The center can be written as $c = \sum_{i=1}^{m} \lambda_i p_i$ with the λ_i being dependent on r^2. Plugging this definition of c into Equation (2) yields

$$z^T z = (c - p_1)^T (c - p_1) = (\sum_{i=1}^{m} \lambda_i p_i - p_1)^T (\sum_{i=1}^{m} \lambda_i p_i - p_1) = r^2 s_{p_1}^2. \qquad (5)$$

s_{p_1}, p_1, and the p_i's are known and the λ_i are dependent on r^2; we can conclude that above equation is a doubly quadratic equation in r. We choose r minimal under the condition that $\forall i : \lambda_i \geq 0$ and $\sum \lambda_i = 1$. With $MB(V) = MB(V, V)$ and Lemma 3.1, we know that if V is a basis, there is such a pair (λ, r) and with Lemma 3.2, we know that there is only one such pair.

If V is, contrary to our assumption, not a basis, then we will not be able to find a pair (λ, r) with $r > 0$, $\forall i : \lambda_i \geq 0$, and $\sum \lambda_i = 1$ as otherwise, V were a basis. □

With the help of Lemma 5.4, the implementation of the operation *violates* is easy. For a point set V, we compute c and r as described in Lemma 5.4. If V is *not* a basis, we will not be able to find c and r according to our constraints; otherwise, we get c and r.

Assuming that the points $p \in P$ have rational coordinates, it follows with Equation 5 being a quadratic equation that r is of the form $a + b\sqrt{d}$ for $a, b, d \in \mathbb{Q}$. c can be written as $c = \sum \lambda_i p_i$ with the λ_i being dependent on r whereas it follows that c is of the form $a_c + b_c\sqrt{d}$. We can conclude that the violates test can be done entirely in rational arithmetic assuming that the point to be tested is represented in rational arithmetic.

5.3 Points with Anisotropic Distance Functions

In [23], Labelle defines a setting where every point in $\mathbb{R}^d$ has a distance function

$$d_p(c) = \sqrt{(p - c)^T M_p (p - c)}$$

for a symmetric positive definite matrix M_p. A symmetric positive definite matrix M_p is a matrix so that $\forall x \in \mathbb{R}^n, x \neq 0 : \ x^T M_p x > 0$ and $M_p^T = M_p$. The condition that M_p is a positive definite matrix ensure that the unit circle $\{x \in \mathbb{R}^d \mid d_p(x) \leq 1\}$ is mapped to an ellipsoid. We show that an ellipsoid is a strictly convex object and therefore $d_p(c)$ fits into our framework.

Note that this definition could be extended to $x \in \mathbb{C}^d$. Looking at the miniball problem in $\mathbb{C}$ is possible because for $x \in \mathbb{C}$, $x^T x$ is the Euclidean distance of the point x from the origin as seen in $\mathbb{R}^2$. It follows that the miniball problem in $\mathbb{C}^d$ can be translated to the miniball problem in $\mathbb{R}^{2d}$.

The setting that we presented in Section 5.2 is a special case of the setting considered here. The reason for handling the scaled case separately is that the implementation of the *violates* operation can be done nicely in the scaled case but not in the anisotropic case. Setting $M_p = I$ for all $p \in P$, we realize that the traditional miniball problem as presented in [37] is a special case of the anisotropic setting.

First, we show that the unit circle in the anisotropic setting is a strictly convex object and second, we present a way to implement the *violates* operation.

Strict Convexity of Level Sets

Lemma 5.5. *The distance function*

$$d_p(c) = \sqrt{(p-c)^T M_p (p-c)}$$

has strictly convex level sets.

Proof. We show that $d_p(c) = \sqrt{(p-c)^T M_p (p-c)}$ is a strictly quasi-convex distance function and with Lemma 2.2, this implies that the level set is strictly convex. A distance function is strictly quasi-convex if

$$\forall x, y \in \mathbb{R}^d, x \neq y, \forall \lambda \in (0,1) : \ d_p((1-\lambda)x + \lambda y) < \max\{d_p(x), d_p(y)\}$$

holds. As we can apply a matrix diagonalization for M_p, we can assume that the space can be rotated so that the ellipsoid described by M_p can be written as

$$d_p(c) = d_{p'}(c') = \sqrt{\sum \frac{(p'_i - c'_i)^2}{a_i}}$$

for some p' and c'. Further,

$$d_{p'}(c') = \sqrt{\sum \frac{((p'_i - y_i) - (c'_i - y_i))^2}{a_i}} = d_{p'-y}(c' - y)$$

holds, we can conclude that the distance function is given as

$$d_p(c) = d_0(c') = \sqrt{\sum \frac{c_i'^2}{a_i}},$$

and that $bd_0(c') = d_0(bc')$ holds. We assume that

$$x, y \in U = \{x \in \mathbb{R}^d \mid \sqrt{\sum \frac{c_i^2}{a_i}} \leq 1\},$$

and the condition for strict quasi-convexity can be reformulated as $\forall \lambda \in (0, 1),\ x, y \in U :$

$$\sqrt{\sum \frac{((1-\lambda)x_i + \lambda y_i)^2}{a_i}} < \max \left\{ \sqrt{\sum \frac{x_i^2}{a_i}}, \sqrt{\sum \frac{y_i^2}{a_i}} \right\}.$$

Assume that $x, y \in U$, $x \neq y$, and as we can scale, we can assume that $d_0(x) = 1$ and $d_0(y) \leq 1$. From the proof of Lemma 5.1, we know that the function $|x|^2 = x^2$ is strictly convex for $x_i \neq y_i$. With $x \neq y$, we know that there has to be an i so that $x_i \neq y_i$. It follows that

$$\begin{aligned}\sqrt{\sum \frac{((1-\lambda)x_i + \lambda y_i)^2}{a_i}} &< \sqrt{(1-\lambda) \sum \frac{(x_i)^2}{a_i} + \lambda \sum \frac{(y_i)^2}{a_i}} \\ &\leq 1 \\ &= d_0(x) \\ &= \max\{d_0(x), d_0(y)\}\end{aligned}$$

and strict quasi-convexity for $d_p(c)$ is shown. We can conclude that the level sets for the distance functions $d_p(c)$ are strictly convex. □

Violates Operation

It remains the implementation of the *violates* operation. In Section 5.2, we have shown that for the scaled setting, the implementation of the *violates* operation boils down to solving a *linear* system of equations. For the anisotropic setting, we will argue that in the general case, it is not possible to reduce the *violates* operation to solving a linear system of equations; the system at hand will be *quadratic*. Having a quadratic system of equations does not make the implementation of the *violates* operation impossible but slows it down and might make it difficult to do exact computations.

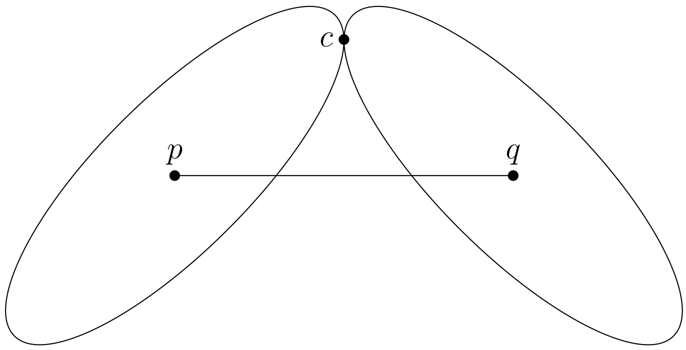


Figure 6: Counter example showing that in the anisotropic setting, the center c has not to be contained in the convex hull of the points in $P = \{p, q\}$.

It seems to be difficult to use geometric properties such as in [16] because the miniball in the anisotropic setting is not a nice geometric object but rather a combinatorial structure. The miniball is defined by the intersection of convex objects but it is *not* defined as a nice object with all basis points contained on its boundary; notice that this is true in general for SC point sets.

Notice that Lemma 5.2 cannot be adapted for the anisotropic setting; Figure 6 gives a counter example.

Lemma 5.6. *Let $V \subseteq P$ be a set of points with distance function*

$$d_p(c) = \sqrt{(p-c)^T M_p (p-c)}$$

and assume that V is a basis of P. Then $MB(V, V) = MB(P)$, where $MB(V, V)$ denotes the miniball containing the set V so that all points in V lie on the boundary of the miniball. $MB(P)$ can be computed in time $O(t(d))$ whereas $t(d)$ is the time that is needed to solve a quadratic system of equations.

Proof. The proof for $MB(V, V) = MB(P)$ can be established in the same way as in Lemma 5.4; we omit this here.

To prove that we can compute $MB(P)$ in time $O(t(d))$, we assume that $V = \{p_1, ..., p_m\}$ for $m \leq d + 1$. The ball $MB(V, V)$ has center c and radius r if and only if $r \geq 0$ and $(p_i - c)^T M_{p_i} (p_i - c) = r^2$ for all i.

The listed conditions give us a system of equations with quadratic occurrences of c. In Lemma 5.4, we were able to remove those quadratic occurrences; let us give an example showing that this is not possible in the anisotropic setting.

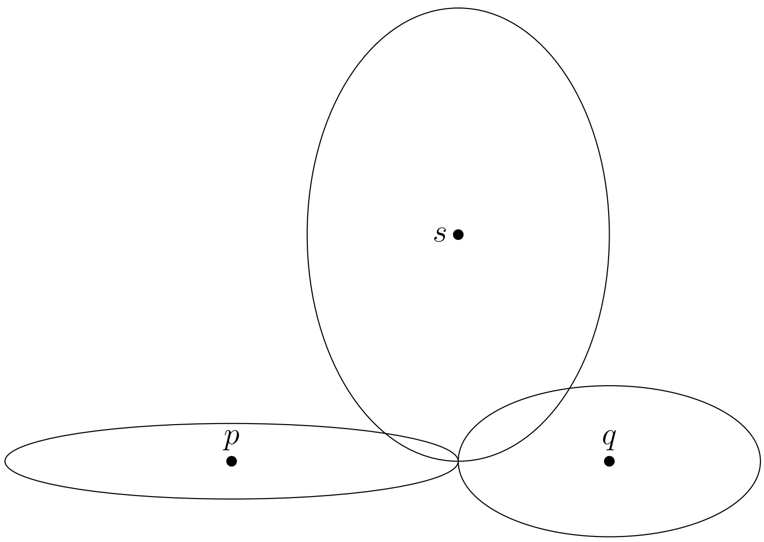


Figure 7: Point set $P = \{p, q, s\}$ with all points in P having positive definite matrices defining the depicted ellipses.

Given a point set $P = \{p, q, s\}$ with $p = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $q = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$, $s = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$ and positive definite matrices

$$M_p = \begin{pmatrix} 3 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \; M_q = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \; M_p = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}.$$

These points and matrices define the configuration given in Figure 7. To find the smallest enclosing ball in the anisotropic setting, we have to solve the system of equations

$$\begin{aligned} d_p(c) &= r \\ d_q(c) &= r \\ d_s(c) &= r. \end{aligned}$$

Rewriting and squaring this yields

$$\begin{aligned} 3p_1^2 - 6p_1c_1 + 3c_1^2 + \frac{1}{2}p_2^2 - p_2c_2 + \frac{1}{2}c_2^2 &= r^2 \\ 2q_1^2 - 4q_1c_1 + 2c_1^2 + q_2^2 - 2q_2c_2 + c_2^2 &= r^2 \\ 2s_1^2 - 4s_1c_1 + 2c_1^2 + 3s_2^2 - 6s_2c_2 + 3c_2^2 &= r^2 \end{aligned}$$

and from this system of equations, we would like to remove all quadratic occurrences of c_1 and c_2. As we can choose the point set arbitrarily, we have to be able to remove the quadratic occurrences for arbitrary c_1 and c_2 which is equivalent to

$$(3a - 2b)c_1^2 + (\frac{1}{2}a - b)c_2^2 = 0$$
$$(3d - 2e)c_1^2 + (\frac{1}{2}d - 3e)c_2^2 = 0$$

having a nontrivial solution for a, b, d, e for all possible c_1 and c_2. It is easy to see that there is no non-trivial solution and it follows that for an arbitrary c, we cannot remove all quadratic occurrences of c. It follows that computing the smallest enclosing ball boils down to solve the quadratic system of equations

$$\forall i : (c - p_i)^T M_{p_i}(c - p_i) = r^2 \tag{6}$$

If V is a basis of P, we know by existence and uniqueness of $MB(P)$ that there is exactly one solution fulfilling this system of equations and minimizing $r > 0$ for $|V| \geq 2$. With a standard method such as [25], we can find this solution. If V is *not* a basis of P, then the system given in (6) will not have a solution with $c \in \mathbb{R}^d$ and $r > 0$. □

As long as we can find the solution for a quadratic system of equations accurately, we are able to compute the miniball in the anisotropic setting exactly. Notice that as long as M_p is a positive definite matrix, it can be chosen arbitrarily.

On a side note, from the proof of Lemma 5.6 it follows immediately that if all the points use the same matrix M_p, then we can remove the quadratic occurrence of c and we are left with solving a linear system of equation. This is intuitively correct as for points that use the same distance function, we can *distort* the space according to the distance function and are left with the miniball problem for the Euclidean setting.

5.4 Smallest Enclosing Ball of Balls

Fischer presents in [12] the problem of finding the smallest enclosing ball of balls. Some of our approaches are taken from Fischer's work and we show that the smallest enclosing ball of balls problem fits into our framework. In [12], Fischer shows how to compute the smallest enclosing ball of balls for small instances (Lemma 5.2 in [12]) and we do not list this here; all we do is to show that the distance function used in the smallest enclosing ball of balls problem is a strictly convex distance function.

Lemma 5.7. *We consider the smallest enclosing ball of balls problem for balls with* $r_p \in \mathbb{R}_{\geq 0}$*. Assume that a ball with radius* r_p *is centered at* $p \in P$*. Then the smallest enclosing ball problem can be formulated with the help of a distance function* $d_p(c) = \|p - c\|_2 + r_p$*.*

Proof. This property is best proved using an image; see Figure 8.

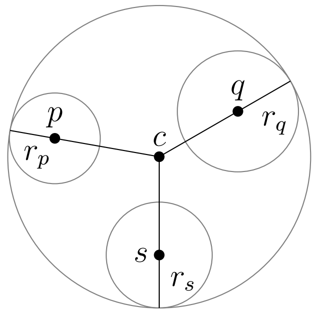


Figure 8: The *Smallest Enclosing Ball of Balls* problem makes use of the distance function $d_p(c) = \|p - c\|_2 + r_p$.

□

With the help of Lemma 5.1, it follows directly that for $r_p \geq 0$,

$$d_p(c) = \|p - c\|_2 + r_p$$

is a strictly convex function and therefore the smallest enclosing ball of balls problem fits into the concept of SC point sets.

For signed balls, Fischer introduces in Lemma 5.4 of [12] a *slider* or *shrinking* operation. The idea is to change the radius of all balls by the same real number and he then shows that such a changed setting has the same basis and, up to the slide-factor, the same miniball $MB(V)$. We can conclude that the problem of the smallest enclosing ball of signed balls can always be transformed to an instance of smallest enclosing ball of balls and it is therefore as well contained in the family of SC point sets. Transforming signed balls to positive balls ensures that the distance function $d_p(c) = \|p - c\|_2 + r_p$ is positive for all $c \in \mathbb{R}^d$ though we could as well argue directly for signed balls; see Section 5.6 for a possible strategy.

5.5 Smallest Enclosing Bregman Balls

In 1967, L. M. Bregman introduced in [3] the concept of Bregman divergence; Censor and Lent summarize in [7] the material that was presented in the original paper. Let us give the definition for the Bregman divergence and show that this distortion measure has strictly convex level sets.

The Bregman divergence

$$D_\phi(p,q) = \phi(p) - \phi(q) - \langle p-q, \nabla_\phi(q)\rangle$$

is constructed with the help of the Bregman function $\phi(p)$ that is defined as follows:

Definition 5.8. ***(Adapted Definition 1.2 from [6])*** *Let S be a non-empty, open, and convex set, and for Λ and the closure $\mathrm{cl}(S)$ of S, $\mathrm{cl}(S) \subseteq \Lambda$ holds. A function $\phi : \Lambda \subseteq \mathbb{R}^n \to \mathbb{R}$ is called a Bregman function if it satisfies the following conditions:*

(i) *ϕ is continuous on $\mathrm{cl}(S)$;*

(ii) *ϕ is strictly convex on $\mathrm{cl}(S)$;*

(iii) *ϕ is differentiable on S;*

(iv) *If $x \in \mathrm{cl}(S)$ and $\alpha > 0$; then the partial level sets $\{z \in S \mid D_\phi(x,z) \leq \alpha\}$ are bounded;*

(v) *If $\{x^k\}_{k\in\mathbb{N}} \subset S$ is a convergent sequence with the limit $x^* \in bdS := \mathrm{cl}(S) \setminus S$, the following limit exists and we have*

$$\lim_{k\to\infty} \langle \nabla\phi(x^k), x^* - x^k\rangle = 0.$$

The class of Bregman divergences [31] includes the L_2 norm $D_\phi(p,q) = \|p-q\|_2^2$, the *Kullback–Leibler divergence*

$$D_\phi(p,q) = \sum_{i=1}^{d} p_i \log\left(\frac{p_i}{q_i}\right) - p_i + q_i,$$

the *Itakura–Saito* distance

$$D_\phi(p,q) = \sum_{i=1}^{d} \frac{p_i}{q_i} - \log\left(\frac{p_i}{q_i}\right) - 1,$$

the *Mahalanobis distance*

$$D_\phi(p,q) = (p-q)^T A (p-q)$$

for a positive definite matrix A, and

$$D_\phi(p,q) = \sum_{i=1}^{d} \frac{p_i^s}{s} + \frac{(s-1)q_i^s}{s} - p_i q_i^{s-1},\ s \in \mathbb{N} \setminus \{0,1\}.$$

Based on the Bregman divergence, we can define smallest enclosing balls in at least three different ways:

i. $d_p(c) = D_\phi(c,p)$

ii. $d_p(c) = D_\phi(p,c)$

iii. $d_p(c) = \frac{D_\phi(c,p)+D_\phi(p,c)}{2}$

Let us check the three different possibilities separately and start with the first case. If we can show that for a fixed $p \in S$, $D_\phi(c,p)$ is a strictly convex function in c, then it follows that a point set P with distance functions $d_p(c) = D_\phi(c,p)$ fits into our setting. Proving strict convexity for $D_\phi(c,p)$ in c for a fixed p and $c \in \mathrm{cl}(S)$ is easy:

$$\begin{aligned}
D_\phi((1-\lambda)x+\lambda y, p) &= \phi((1-\lambda)x+\lambda y) - \langle(1-\lambda)x+\lambda y, \nabla_\phi(p)\rangle \\
&\quad -\phi(p) + \langle p, \nabla_\phi(p)\rangle \\
&= \phi((1-\lambda)x+\lambda y) - (1-\lambda)\langle x, \nabla_\phi(p)\rangle \\
&\quad -\lambda\langle y, \nabla_\phi(p)\rangle - \phi(p) + \langle p, \nabla_\phi(p)\rangle \\
&< (1-\lambda)(\phi(x) - \langle x, \nabla_\phi(p)\rangle - \phi(p) + \langle p, \nabla_\phi(p)\rangle) \\
&\quad +\lambda(\phi(y) - \langle y, \nabla_\phi(p)\rangle - \phi(p) + \langle p, \nabla_\phi(p)\rangle \qquad (7) \\
&= (1-\lambda)D_\phi(x,p) + \lambda D_\phi(y,p)
\end{aligned}$$

The inequality in Equation 7 follows from the strict convexity of $\phi(x)$ for $x \in \mathrm{cl}(S)$. It remains to show that for a point set $P \subseteq S$ with S as introduced in Definition 5.8, $c \in \mathrm{cl}(S)$ has to hold in the general case. In [32], Nock and Nielsen state that for the center c, there are α_i with $\sum_{i=1}^{n} \alpha_i = 1$ so that

$$c = \nabla_\phi^{-1}\left(\sum_{i=1}^{n} \alpha_i \nabla_\phi(p_i)\right).$$

Using the fact that $\phi(x)$ is strictly convex and therefore $\nabla_\phi(x)$ is bijective, they conclude that c has to lie in $\mathrm{cl}(S)$. It follows that we can reformulate the miniball problem as

$$\begin{array}{ll}
\text{minimize} & r \\
\text{subject to} & \forall p \in P:\ d_p(c) \le r \\
 & c \in \mathrm{cl}(S).
\end{array}$$

without changing its solution. It is important to note that this does *not* directly imply that $d_p(c) = D_\phi(c, p)$ fits into our framework as we have shown strict convexity of $d_p(c)$ only on the set $\mathrm{cl}(S)$. We have to show that we can define the level sets $L(p, t)$ on a convex space $\mathrm{cl}(S) \subset \mathbb{R}^d$ and that none of the properties that we have shown for $\mathbb{R}^d$ is destroyed by restricting to $\mathrm{cl}(S)$. The definition of the level set $L(p, t)$ can be adapted as

$$L(p, t) = \{x \in \mathrm{cl}(S) \mid d_p(x) \leq t\}$$

and for the strict convexity of the level set, we select points $x, y \in \mathrm{cl}(S)$; consult Section 2. For the definition of strictly quasi–convex $d_p(c)$, we select again points $x, y \in \mathrm{cl}(S)$ and it should be possible to prove an adapted version of Lemma 2.2. Existence of the miniball follows analogously as in Lemma 3.1 and using the knowledge that any possible center c has to be contained in $\mathrm{cl}(S)$, it should be possible to show uniqueness though the proof strategy as used in Lemma 3.2 might have to be slightly tweaked. The proof of the combinatorial dimension works as well as $\mathrm{cl}(S)$ is a convex object. Last, the proof for the miniball problem being an LP-type problem can be applied directly as all preconditions have been established. Although this analysis is not formal, we proceed with the assumption that the distance function $d_p(c) = D_\phi(c, p)$ fits into our framework.

For the second case with

$$d_p(c) = D_\phi(p, c) = \phi(p) - \phi(c) - \langle p - c, \nabla_\phi(c)\rangle,$$

$-\phi(c)$ is a strictly *concave* function but at the same time $\nabla_\phi(c)$ is not a constant. It might be possible to find a $\phi(c)$ so that $D_\phi(p, c)$ is strictly convex or at least strictly quasi-convex in c but this seems not to hold for the general case.

For the third case,

$$\begin{aligned} d_p(c) &= \frac{D_\phi(c, p) + D_\phi(p, c)}{2} \\ &= \frac{\langle c - p, \nabla_\phi(p)\rangle + \langle p - c, \nabla_\phi(c)\rangle}{2} \end{aligned}$$

and it is not obvious whether strict convexity holds; we assume that it does not hold for either argument in the general case.

Only the Bregman balls of the first type seem to be fitting into our setting. Actually, Nock and Nielsen consider this setting and presented exact [31] and approximate [32] solutions. In [31], they claim that Welzl's algorithm (Algorithm 1) can be applied to Bregman balls though they do not show that Lemma 4.1 indeed holds.

Let us show that Lemma 4.1 holds for a class of distance functions with the first type Bregman ball being part of that class. To do so, we first show two properties. We consider the class of strictly convex functions $d_p(c)$ so that the system of equations

$$d_q(c) = r, \ \forall q \in Q$$

can be reformulated as a linear system of equations. This can be clearly done for the first type Bregman ball because

$$d_q(c) = D_\phi(c, q) = \phi(c) - \phi(q) - \langle c - q, \nabla_\phi(q)\rangle = r, \ \forall q \in Q.$$

This can be reformulated with $t = r - \phi(c)$ as

$$\langle q, \nabla_\phi(q)\rangle - \phi(q) - \langle c, \nabla_\phi(q)\rangle = t, \ \forall q \in Q.$$

Because $q \in Q$ is a constant, this is a system of linear equations. Further, for $b_1, b_2 \in \mathrm{cl}(S)$ and $\lambda \in (0, 1)$, define

$$\begin{aligned} e(b_1, b_2, \lambda) &= (1-\lambda)d_p(b_1) + \lambda d_p(b_2) - d_p((1-\lambda)b_1 + \lambda b_2) \\ &= (1-\lambda)\phi(b_1) + \lambda\phi(b_2) - \phi((1-\lambda)b_1 + \lambda b_2) \\ &> 0. \end{aligned}$$

Observe that the last inequality holds with strict convexity for ϕ and that $e(b_1, b_2, \lambda)$ is not dependent on p.

Lemma 5.9. *Let $Q \subseteq P$ with $Q \neq \emptyset$ be point sets and assume that $d_p(c)$ is the distance function for all $p \in P$. Define $MB(P, Q)$ to be the smallest ball enclosing P and containing Q on its boundary and assume that $MB(P, Q)$ exists. Assume further that the system of equations $\forall q \in Q : d_q(c) = r$ can be reformulated as a system of linear equations and that for all $\lambda \in (0, 1)$,*

$$e(b_1, b_2, \lambda) = (1-\lambda)d_p(b_1) + \lambda d_p(b_2) - d_p((1-\lambda)b_1 + \lambda b_2) > 0$$

exists and is not dependent on p. Then $MB(P, Q)$ is unique.

Proof. Finding the smallest enclosing ball containing Q on its boundary but not necessarily containing P can be formulated as finding a minimum r fulfilling

$$d_q(c) = r, \ \forall q \in Q.$$

According to our precondition, there are $d'_q(c)$ and t so that the system of equations can be reformulated as

$$d'_q(c) = t, \ \forall q \in Q.$$

This is a system of linear equations $Ax = b$ with $x_i = c_i$ for $i \in \{1, ..., d\}$ and $x_{d+1} = t$. As by our precondition, we know that there is a smallest enclosing ball containing Q on its boundary and therefore this system of equations has at least one solution.

If the system of linear equations has *exactly one* solution, then this solution defines the unique smallest enclosing ball with Q on the boundary. By precondition, we know that $MB(P, Q)$ exists and we conclude that the ball defined by the only existing solution of $Ax = b$ contains all $p \in P$; we are done for this case.

So let us assume that $Ax = b$ has infinitely many solutions. The solution space for $Ax = b$ is defined as the affine space $L_1 = x' + L_0$ with x' any solution for $Ax = b$ and L_0 the solution space for $Ax = 0$. Assume that $f : \mathbb{R}^{d+1} \rightarrow \mathbb{R}^d$ is a projection with $f(x) = y$ so that $x_i = y_i$ for all $i \in \{1, ..., d\}$. Let $L'_1 = f(L_1)$; L'_1 is the solution space ignoring the value of t. This can be done because t is uniquely determined by the center c and the points in Q.

The space L'_1 contains all possible centers that have the same distance to all points in Q. Let us define $C_Q(r) = \bigcap_{q \in Q} L(q, r)$ and observe that $C_Q(r)$ is a strictly convex object. $C_Q(r)$ defines the set of all centers so that the ball with center c and radius r contains all points in Q but notice that the points in Q are not necessarily on the boundary of the ball.

Because of the existence of $MB(P, Q)$, we know that L'_1 and $C_Q(r)$ have to intersect for some r and this intersection is a strictly convex object; let us denote this intersection as $C'_Q(r) = C_Q(r) \cap L'_1$. The boundary of $C'_Q(r)$ contains all centers c so that the ball with center c and radius r contains all points in Q on its boundary.

Let us analogously define $C_P(r) = \bigcap_{p \in P} L(p, r)$ and $C'_P(r) = C_P(r) \cap L'_1$. Because $C_P(r)$ is a strictly convex object and L'_1 is an affine subspace, there is a smallest r_0 so that $C'_P(r_0) \neq \emptyset$. Assume that $b = C'_P(r_0)$ is the unique point contained in that set; the point has to be unique because a hyperplane and a strictly convex object first intersect in exactly one point. Notice that $C'_Q(r_0) \neq \emptyset$ has to hold because $C'_P(r_0) \subseteq C'_Q(r_0)$.

If the point b lies on the boundary of $C'_Q(r_0)$, then we have found the unique smallest ball enclosing all points in P and containing all points in Q on the boundary. This holds because $\forall q \in Q : d_q(b) = r_0$ and $\forall p \in P : d_p(b) \leq r_0$.

So assume that b is properly contained in $C'_Q(r_0)$. Assume that $r_1 > r_0$ is the first radius so that the boundary of $C'_Q(r_1)$ and $C'_P(r_1)$ intersect. If they intersect only in one point, then we have found the unique center of $MB(P, Q)$ and we are done. Therefore assume that for r_1 as small as possible with $r_1 > r_0$, there are at least two points b_1, b_2 on the boundary of $C'_P(r_1)$ so that both b_1 and b_2 are as well contained on the boundary of $C'_Q(r_1)$; this are the conditions for $MB(P, Q)$ to be *not* unique. Based on those assumptions, let us show a contradiction and conclude that $MB(P, Q)$ is unique. From the given conditions, it follows that

$$\forall q \in Q : d_q(b_1) = d_q(b_2) = r_1$$

and

$$\forall p \in P : d_p(b_1) \leq r_1 \wedge d_p(b_2) \leq r_1.$$

We know that

$$\lambda \in (0, 1) : e(b_1, b_2, \lambda) = (1 - \lambda)d_p(b_1) + \lambda d_p(b_2) - d_p((1 - \lambda)b_1 + \lambda b_2) > 0$$

exists and is not dependent on p. For all $p \in P$ and $\lambda \in (0, 1)$, we have

$$(1 - \lambda)d_p(b_1) + \lambda d_p(b_2) \quad = e(b_1, b_2, \lambda) + d_p((1 - \lambda)b_1 + \lambda b_2)$$

It follows with $Q \subseteq P$ that

$$\forall q \in Q : d_q((1 - \lambda)b_1 + \lambda b_2) = r_1 - e(b_1, b_2, \lambda) < r_1$$

and

$$\forall p \in P : d_p((1 - \lambda)b_1 + \lambda b_2) \leq r_1 - e(b_1, b_2, \lambda) < r_1.$$

This means that for $r_1 - e(b_1, b_2, \lambda) < r_1$, $C'_Q(r_1 - e(b_1, b_2, \lambda))$ and $C'_P(r_1 - e(b_1, b_2, \lambda))$ intersect in at least one point; a contradiction to the assumption that r_1 is minimal and uniqueness for $MB(P, Q)$ follows. □

Lemma 5.9 allows us to conclude that Algorithm 1 can indeed be applied. For the case of Bregman balls with points $p \in P$ having their own *individual* basis function $\phi_p(x)$, it is easy to find a counter example showing that this Bregman ball cannot be found with Welzl's algorithm as given in Algorithm 1. Notice that for $\phi_p(x) = s_p\|x\|_2^2$, we have the Bregman divergence

$$\begin{aligned} d_p(c) = D_{\phi_p}(c, p) &= \phi_p(c) - \phi_p(p) - \langle c - p, \nabla_{\phi_p}(p)\rangle \\ &= s_p\|c - p\|_2^2. \end{aligned}$$

This is almost the same setting as we used to construct the counter example in Figure 3 and the original counter example can be nicely adapted. Alternatively, we could use the Mahalanobis distance with matrices A_p and this results in the well-known anisotropic setting.

Even though we can apply Algorithm 1, let us proceed by showing how to implement the *violates* operation for the first type of Bregman balls.

Violates Operation

The definition of the Bregman divergence $d_p(c) = D_\phi(c, p)$ looks a bit complicated but keeping in mind that p is constant, we know that both $\nabla_\phi(p)$ and $r_p = -\phi(p) + \langle p, \nabla_\phi(p) \rangle$ are constant.

Lemma 5.10. *Let $V \subseteq P$ be a set of points with distance function*

$$d_p(c) = D_\phi(c, p) = \phi(c) - \phi(p) - \langle c - p, \nabla_\phi(p) \rangle$$

and assume that V is a basis of P. With $MB(V, V)$, let us denote the set of miniballs of the point set V that contains V on its boundary. Then $MB(V, V) = MB(P)$ and $MB(P)$ can be computed in time $O(t(d))$ where $O(t(d))$ is the time used to solve an arbitrary system of equations of size d.

Proof. The first part of the proof can be established in the same way as for Lemma 5.4.

Above, we have observed that for a constant p, $\langle c, \nabla_\phi(p) \rangle$ is linear in c and r_p is constant. It follows that if $\phi(c)$ is linear in c, we get a linear system of equations and can proceed analogously to Lemma 5.4. If $\phi(c)$ is of polynomial form, we can proceed analogous to Lemma 5.6. In case $\phi(c)$ is not of polynomial form, we have to solve an arbitrary system of equations and we assume that this can be done in time $O(t(d))$. □

5.6 Superorthogonal Balls

In [12], Fischer presents the concept of superorthogonal balls. Such balls are defined for a point set P having points $p \in P$ with distance function

$$d_p(c) = \|p - c\|_2^2 - r_p^2.$$

With Lemma 5.1, it follows immediately that $d_p(c)$ has a strictly convex level set $L(p, t)$. It is important to notice that there are points $x \in \mathbb{R}^d$ with $d_p(x) < 0$. This prohibits that $d_p(c)$ is a proper distance function but we can define

$$d'_p(c) = \|p - c\|_2^2 - r_p^2 + \max_{p \in P}\{r_p^2\}$$

so that $d_p(x) \geq 0$ for all $x \in \mathbb{R}^d$ and for all $p \in P$. $d'_p(c)$ is a proper distance function with a strictly convex level set $L'(p, t)$. We can solve the smallest enclosing ball problem for the distance function $d'_p(c)$; let us assume that the smallest enclosing ball has center c and radius r. We claim that the ball with center c and radius $r - \max_{p \in P}\{r_p^2\}$ is the smallest enclosing ball for the point set P using the distance function $d_p(c)$.

Assume not and that there is a ball enclosing P for the function $d_p(c)$ with radius $r' < r - \max_{p\in P}\{r_p^2\}$; for all points $p \in P$, $d_p(c) \leq r'$ has to hold. From this, it follows that there is a ball with radius $r' + \max_{p\in P}\{r_p^2\} < r$ enclosing all points in P for the distance function $d'_p(c)$; a contradiction to our assumption that the smallest enclosing ball has radius r.

Let us show how to implement the *violates* operation for superorthogonal balls.

Violates Operation

The function $d'_p(c)$ can be written as

$$d'_p(c) = \|p - c\|_2^2 - r_p^2 + \max_{p\in P}\{r_p^2\} = f_p(\|p - c\|_2)$$

for $f_p(x) = x^2 - r_p^2 + \max_{p\in P}\{r_p^2\}$. The function $f_p(x)$ is a strictly increasing function for $x \in \mathbb{R}_{\geq 0}$ and Lemma 5.2 can be applied; the center c of the smallest enclosing ball is contained in the convex hull of the point set P. Lemma 5.3 is solely based on Lemma 5.2 and therefore holds as well. Our next Lemma follows closely Lemma 5.4:

Lemma 5.11. ***(Adaption of Lemma 5.4)*** *Let $V \subseteq P$ be a set of points with distance function*

$$d'_p(c) = \|p - c\|_2^2 - r_p^2 + \max_{p\in P}\{r_p^2\}$$

and assume that V is a basis of P. With $MB(V, V)$, let us denote the set of miniballs of the point set V that contains V on its boundary. Then $MB(V, V) = MB(P)$ and $MB(P)$ can be computed in time $O(d^3)$.

Proof. We proceed along the same line as in the proof of Lemma 5.4. The first part can be done equivalently, so let us show how to compute $MB(V, V)$.

Assume that $V = \{p_1, ..., p_m\}$ for $m \leq d + 1$ and that the ball $MB(V, V)$ has center c and radius $r \geq 0$. This hold if and only if $\|p_i - c\|_2^2 = r + r_{p_i}^2 - \max_{p\in P}\{r_p^2\}$ for all i. Defining $z = c - p_1$ and $z_{p_i} = p_i - p_1$ for $1 < i \leq m$, we can reformulate these conditions as

$$\begin{aligned} z^T z &= r + r_{p_1}^2 - \max_{p\in P}\{r_p^2\} \\ (z_{p_i} - z)^T (z_{p_i} - z) &= r + r_{p_i}^2 - \max_{p\in P}\{r_p^2\},\ i \in [m] \setminus \{1\} \end{aligned}$$

Subtracting the former from the later yields $m - 1$ equations of the form

$$z_{p_i}^T z_{p_i} - 2z^T z_{p_i} = r_{p_i}^2 - r_{p_1}^2,\ i \in [m] \setminus \{1\}. \tag{8}$$

Notice that those equations are linear in z. From Lemma 5.2, we know that c can be written as $c = \sum \lambda_i p_i$ for λ_i with $\sum \lambda_i = 1$ and it follows that

$$z = c - p_1 = \sum_{i=1}^{m} \lambda_i p_i - p_1 = \sum_{i=2}^{m} \lambda_i p_i - \sum_{i=2}^{m} \lambda_i p_1 = \sum_{i=2}^{m} \lambda_i (p_i - p1) = Q\lambda$$

for $Q = (z_{p_2}, ..., z_{p_m})$ and $\lambda = (\lambda_2, ..., \lambda_m)$. This equation can be used to rewrite Equation 8 as

$$2z_{p_i}^T Q\lambda = z_{p_i}^T z_{p_i} - r_{p_i}^2 + r_{P_1}^2,\ i \in [m] \setminus \{1\}. \tag{9}$$

We can conclude that this is a linear system of equations of the form $A\lambda = b$ with $A = 2Q^TQ$. In the same way as in Lemma 5.4, we can conclude that Q^TQ is regular. The linear system of equations can be solved by a standard method in $O(d^3)$. We select a solution so that $\lambda_i \geq 0$ for all $i \in [m] \setminus \{1\}$ and $\lambda_1 = 1 - \sum_{i=2}^{m} \lambda_i$; by existence of the smallest enclosing ball, we know that there is one solution fulfilling these conditions. □

Notice that the system of equations

$$d_q(c) = \|q - c\|_2^2 - r_q^2 = r,\ \forall q \in Q$$

can be reformulated with $t = r - c^2$ as

$$d'_q(c) = q^2 - 2qc - r_q^2 = t,\ \forall q \in Q$$

resulting in a linear system of equations. Further observe that with strict convexity for $d_p(c)$ and $\lambda \in (0, 1)$,

$$\begin{aligned} e(b_1, b_2, \lambda) &= (1-\lambda)d_p(b_1) + \lambda d_p(b_2) - d_p((1-\lambda)b_1 + \lambda b_2) \\ &= (1-\lambda)(p^Tp - 2p^Tb_1 + b_1^Tb_1 - r_p^2) \\ &\quad +\lambda(p^Tp - 2p^Tb_2 + b_2^Tb_2 - r_p^2) \\ &\quad -(p^Tp - 2p^T((1-\lambda)b_1 + \lambda b_2) \\ &\quad +((1-\lambda)b_1 + \lambda b_2)^T((1-\lambda)b_1 + \lambda b_2) - r_p^2) \\ &= (1-\lambda)b_1^Tb_1 + \lambda b_2^Tb_2 - ((1-\lambda)b_1 + \lambda b_2)^T((1-\lambda)b_1 + \lambda b_2) \\ &> 0 \end{aligned}$$

exists and is not dependent on p; it follows that Lemma 5.9 can be applied. We can conclude that additionally to Algorithm 2, Algorithm 1 can as well be used to compute the smallest enclosing superorthogonal ball. This is, to the best of our knowledge, a new insight.

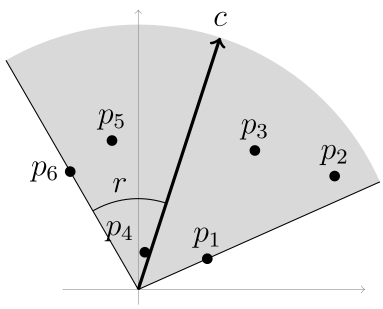


Figure 9: Cone with center c and angle r for the point set $\{p_1, p_2, p_3, p_4, p_5, p_6\}$.

5.7 Smallest Enclosing Cone

Given a set of points P, we would like to find the smallest *cone* with a $d-1$ dimensional ball as base so that the rays from the origin — called apex — to the points P are contained in the cone. This is equivalent to minimizing the angles between a *center vector* and all other vectors in the set and gives us the distance function

$$d_p(c) = \arccos\left(\frac{p^T c}{\|p\|\|c\|}\right).$$

The cone with center vector c and radius — or better angle — r is defined as

$$\{x \in \mathbb{R}^d \mid d_c(x) \leq r\}.$$

We consider only cones that have a radius $r < \frac{\pi}{2}$ and therefore the maximum distance $d_x(y)$ for two points $x, y \in \{x \in \mathbb{R}^d \mid d_c(x) \leq r\}$ is strictly smaller than π. Without this restriction, the smallest enclosing cone might not be unique as it can be seen in Figure 10. It will turn out that without this restriction, $d_p(c)$ cannot have a strictly convex level set.

Before we reduce the smallest enclosing cone problem to the smallest enclosing ball problem, we try to show that the distance function $d_p(c)$ has strictly convex level sets though one bit will be missing.

Lemma 5.12. *Given functions g and f with $dom(g)$ being the domain of g,*

$$a, b \in dom(g) : a < b \Rightarrow g(a) < g(b)$$

and

$$\lambda \in (0,1) : f((1-\lambda)x + \lambda y) < (1-\lambda)f(x) + \lambda f(y).$$

Then $g \circ f$ is a strictly quasi-convex function

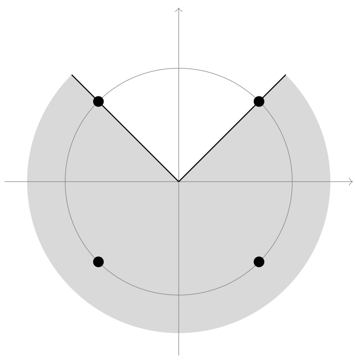


Figure 10: Example for a case with $r \geq \frac{\pi}{2}$ having four smallest enclosing cones; one of the cones is hinted at by the gray area.

Proof. Plugging in yields

$$\begin{aligned} \lambda \in (0,1) : g(f((1-\lambda)x + \lambda y)) &< g((1-\lambda)f(x) + \lambda f(y)) \\ &\leq \max\{g(f(x)), g(f(y))\} \end{aligned}$$

whereas the last inequality holds because $(1-\lambda)f(x)+\lambda f(y)$ describes a line between $f(x)$ and $f(y)$; the maximum value on this line is attained in one of the end points or in all points if $f(x) = f(y)$. $\square$

A similar Lemma can be formulated for the combination of a strictly decreasing and a strictly concave function.

Lemma 5.13. *Given functions g and f with*

$$a, b \in dom(g) : a < b \Rightarrow g(a) > g(b)$$

and

$$\lambda \in (0,1) : f((1-\lambda)x + \lambda y) > (1-\lambda)f(x) + \lambda f(y).$$

Then $g \circ f$ is a strictly quasi-convex function

Proof. We proceed as in Lemma 5.12. Plugging in yields

$$\begin{aligned} \lambda \in (0,1) : g(f((1-\lambda)x + \lambda y)) &< g((1-\lambda)f(x) + \lambda f(y)) \\ &\leq g(\min\{f(x), f(y)\}) \\ &\leq \max\{g(f(x)), g(f(y))\}. \end{aligned}$$

Again, the second to the last inequality holds because $(1-\lambda)f(x)+\lambda f(y)$ describes a line and its minimum is attained at either $f(x)$ or $f(y)$. $\square$

We try to employ Lemma 5.13 to show that $d_p(c)$ is a strictly quasi-convex distance function but unfortunately, one piece is missing. Observe that p is contained in the cone with center c and radius r if and only if $p' = \frac{p}{\|p\|_2}$ is contained in the same cone; assume therefore that we are given a point set $P' = \{p \in P \mid \frac{p}{\|p\|_2}\}$. Let us first try to show that $\frac{p^T c}{\|p\|_2 \|c\|_2}$ is a strictly concave function in c. For $x, y \in \mathbb{R}^d$ with $\|x\| = \|y\|$, we can show strict concavity for all $\lambda \in (0,1)$ as

$$\begin{aligned} (1-\lambda)\frac{p^T x}{\|p\|_2\|x\|_2} + \lambda \frac{p^T y}{\|p\|_2\|y\|_2} &= \frac{(1-\lambda)p^T x + \lambda p^T y}{\|p\|((1-\lambda)\|x\|_2 + \lambda\|y\|_2)} \\ &< \frac{p^T((1-\lambda)x + \lambda y)}{\|p\|_2\|(1-\lambda)x + \lambda y\|_2} \end{aligned}$$

whereas the inequality holds because $\|x\|_2$ is with Lemma 5.1 a strictly convex function. For x, y with $\|x\| \neq \|y\|$, we are unfortunately not able to show the strict concavity though we assume that it holds as well for this case. As a consequence, we proceed with the *assumption* that $\frac{p^T c}{\|p\|_2\|c\|_2}$ is a strictly concave function. If this assumption should turn out to be wrong, then any result following that is based on the strict convexity of the distance function $d_p(c)$ will be void.

The function $\arccos(t)$ is a strictly decreasing function in the interval $t \in (-1, 1)$; this holds because

$$\frac{\partial}{\partial t}\arccos(t) = -\frac{1}{\sqrt{1-t^2}}.$$

With Lemma 5.13, we can conclude — using above assumption — that for $p \in P'$, two vectors x and y with $\frac{x}{\|x\|} \neq \frac{y}{\|y\|}$ and an angle of less than $\frac{\pi}{2}$ to p, and therefore having an angle strictly smaller than π between x and y, we have

$$\arccos\left(\frac{p^T((1-\lambda)x + \lambda y)}{\|(1-\lambda)x + \lambda y\|_2}\right) < \max\left\{\arccos\left(\frac{p^T x}{\|x\|_2}\right), \arccos\left(\frac{p^T y}{\|y\|_2}\right)\right\}$$

and strict quasi-convexity for $d_p(c)$ follows. Notice that $\arccos(t)$ is a strictly decreasing function only in the interval $(-1, 1)$. This is not a problem for

$$t = \frac{p^T((1-\lambda)x + \lambda y)}{\|(1-\lambda)x + \lambda y\|_2} = 1$$

because this implies that $((1-\lambda)x + \lambda y)$ and p point into the same direction. By our assumption, x and y cannot point into the same direction and therefore $\frac{p^T x}{\|x\|_2} < 1$ or $\frac{p^T y}{\|y\|_2} < 1$. This leads to $\arccos\left(\frac{p^T x}{\|x\|_2}\right) > 0$ or $\arccos\left(\frac{p^T y}{\|y\|_2}\right) > 0$ and it follows that strict convexity holds as well for this case.

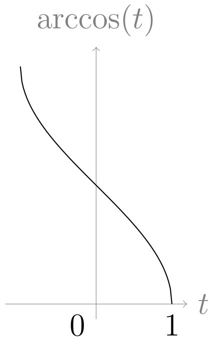


Figure 11: $arccos(t)$ in the interval $[-1, 1]$.

Notice that for a point set P, strict quasi-convexity for $d_p(x)$ holds only if there is a center c that has an angle of less than $\frac{\pi}{2}$ to all points in P. This is equivalent to stating that there has to be a $(d-1)$ dimensional hyperplane through the origin so that all points in P are strictly on one side of the hyperplane; we say that all points lie in one hemisphere. We have seen above that if this does not hold, the smallest enclosing cone might not be unique.

Instead of showing how to implement *violates*, we show that the smallest enclosing cone problem can be reduced to the smallest enclosing *ball* problem. Even though the smallest enclosing cone problem fits into our framework (using above assumption), it makes sense to show this reduction. Our framework gives access to a general purpose algorithm whereas more sophisticated algorithms might exist for the smallest enclosing ball problem; e.g. [14]. The difference between the general purpose and specialized algorithms might be significant; especially in high dimensions.

The reduction of the smallest enclosing cone problem to the smallest enclosing ball problem was first shown by Lawson in [24] though without proof of correctness. Barequet and Elber choose in [1] a different approach and employed a smallest enclosing circle algorithm on the surface of a 3-dimensional sphere.

Reduction to Smallest Enclosing Ball

Let us consider the smallest enclosing cone of a point set $P \subset \mathbb{R}^d$. The smallest enclosing cone is not dependent on the length of the vectors in P and therefore we can set $p' = \frac{p}{\|p\|}$ for all points $p' \neq \mathbf{0}$ and observe that $\mathbf{0}$ is contained in every cone with apex at the origin and we can therefore remove $\mathbf{0}$ from the point set P. Let us call this modified point set P' and notice that this transformation can be done in time $O(dn)$. All points in P' lie on the

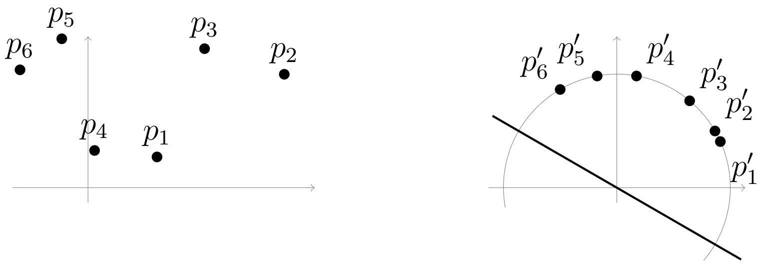


Figure 12: Example for placing the points in P onto the unit sphere. The line in the second example indicates that the points lie in one hemisphere.

unit sphere with the origin as its center and by our precondition, all points in P' have to lie strictly in one hemisphere.

Let us prove the reduction of the smallest enclosing cone problem to the smallest enclosing ball problem with two approaches. The first one is a geometric argumentation and the second one is based on showing the equality of two optimization problems.

For a point set P' lying on the unit sphere, we know that the center c of the smallest enclosing ball is equal to the point in the convex hull of P' that is closest to the origin. This property follows directly from a theorem stated in [26]:

Theorem 5.14. ***(Theorem 8.7.4 from [26])*** *Let $p_1, ..., p_n$ be points in $\mathbb{R}^d$, and let Q be the $d \times n$ matrix whose jth column is formed by the d coordinates of the point p_j. Let us consider the optimization problem*

$$\begin{array}{ll} \textit{minimize} & x^T Q^T Q x - \sum_{j=1}^n x_j p_j^T p_j \\ \textit{subject to} & \sum_{j=1}^n x_j = 1 \\ & x \geq 0 \end{array} \tag{10}$$

in the variables $x_1, ..., x_n$. Then the objective function $f(x) := x^T Q^T Q x - \sum_{j=1}^n x_j p_j^T p_j$ is convex, and the following statements hold.

(i) Problem (10) has an optimal solution x^*.

(ii) There exists a point p^* such that $p^* = Qx^*$ holds for every optimal solution x^*. Moreover, the ball with center p^* and squared radius $-f(\mathbf{x}^*)$ is the unique ball of smallest radius containing P.

For points on the unit sphere, $p_j^T p_j = 1$ and the optimization problem

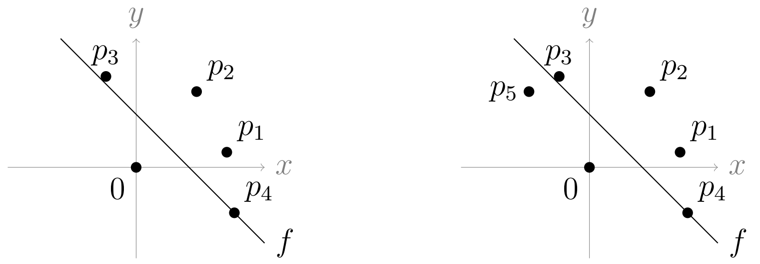


Figure 13: f covers $\{p_1, p_2, p_3, p_4\}$ but does not cover $\{p_1, p_2, p_3, p_4, p_5\}$.

from Theorem 5.14 can be reformulated for our case as

$$\begin{array}{rl} \text{minimize} & x^T Q^T Q x \\ \text{subject to} & \sum_{j=1}^{n} x_j = 1 \\ & x \geq 0 \end{array}$$

The objective function can be interpreted as the distance between the point $p^* = Qx$ and the origin. Due to the additional constraints, the point p^* has to be in the convex hull of the point set and our claim follows.

By our precondition, all points in P' have to lie strictly in one hemisphere and we can conclude that the origin is not contained in the convex hull of P'; therefore, the distance d between the origin and the center c is strictly greater than zero.

Because we minimize the distance between the origin and the center c, we can conclude that the center c has to lie on the boundary of the convex hull of P'. The boundary face g containing c has to be spanned by at least two points; assuming that $|P'| > 1$. The vector c has to be perpendicular to g as otherwise, we could find a point $c' \neq c$ on g that is closer to the origin.

Definition 5.15. *For a $d-1$ dimensional hyperplane f not containing the origin, let us say that f **covers** a point if and only if the point is either on the hyperplane f, or the origin and the point are on opposite sides of f. We say that a point set P is covered by f if and only if all points in P are covered.*

Lemma 5.16. *Given a hyperplane f intersecting the unit sphere and let us assume that the point b on f closest to the origin has $d = \|b\|_2 > 0$. Then for a point set P' lying on the unit sphere, every point covered by f is contained in the ball with center b and radius $r = \sqrt{1-d^2}$.*

Proof. Assume that $Q' \subseteq P'$ is the set of points covered by f. For every point $q' \in Q'$, let $a_{q'}$ be the distance between q' and f; points lying *on* f

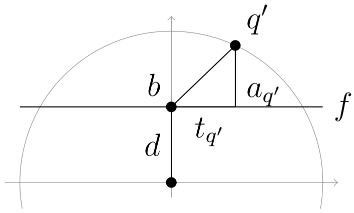


Figure 14: A point q' covered by f is contained in the ball with center b and radius $r = \sqrt{1-d^2}$.

have $a_{q'} = 0$ and all other points in Q' have a positive distance. For every point $q' \in Q'$, consider the 2 dimensional plane spanned by the two vectors q' and b. With Pythagoras, we know that the distance between b and the projection of q' onto f is

$$t_{q'} = \sqrt{1 - (d + a_{q'})} = \sqrt{1 - d^2 - a_{q'}^2 - 2da_{q'}}$$

and it follows that the distance between b and q' is

$$\sqrt{a_{q'}^2 + 1 - d^2 - a_{q'}^2 - 2da_{q'}} = \sqrt{1 - d^2 - 2da_{q'}}.$$

With both $d > 0$ and $a_{q'} \geq 0$, it follows that all points covered by f are contained in the circle with center b and radius $r = \sqrt{1-d^2}$.

□

Next, we will use Lemma 5.16 to show that all points in P' are covered by h.

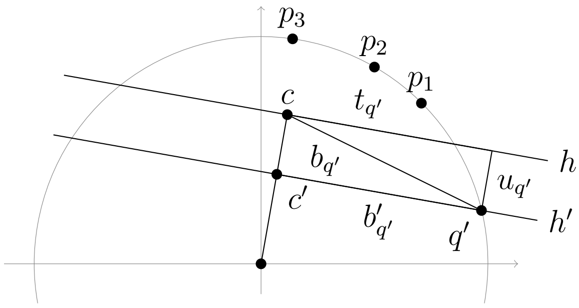
Lemma 5.17. *Assume that the smallest enclosing ball for a point set P' lying on the unit sphere and being contained in one hemisphere has center c and radius r. Further assume that h is the plane perpendicular to the vector c that contains the point c. Then h covers the point set P'.*

Proof. Assume that h does *not* cover P' and that $q' \in P'$ is a point *not* covered by h. Let $u_{q'}$ be the distance between q' and h and assume that we have chosen q' so that $u_{q'}$ is maximal. Because q' is not covered by h, $u_{q'} > 0$ has to hold. Let $t_{q'}$ be the distance between c and the projection of q' onto h;

$$t_{q'} = \sqrt{1 - (d - u_{q'})^2} = \sqrt{1 - d^2 - u_{q'}^2 + 2du_{q'}}.$$

We can conclude that the distance $b_{q'}$ between q' and c is defined as

$$b_{q'} = \sqrt{1 - d^2 - u_{q'}^2 + 2du_{q'} + u_{q'}^2} = \sqrt{1 - d^2 + 2du_{q'}}.$$



Notice that if the ball with center c and radius r covers q', then $r \geq b_{q'}$ has to hold.

Let h' be a $d-1$ dimensional hyperplane that is perpendicular to the vector c and that contains q'; let c' be the intersection of h' and the vector c and let $d' = \|c'\|_2 = d - u_{q'}$. The distance $b'_{q'}$ between q' and c' is defined as

$$b'_{q'} = \sqrt{1 - d'^2} = \sqrt{1 - (d - u_{q'})^2} = \sqrt{1 - d^2 - u_{q'}^2 + 2du_{q'}}$$

which, by geometrical reasoning, can be seen to be the same as $t_{q'}$. It follows immediately that

$$b'_{q'} = \sqrt{1 - d^2 - u_{q'}^2 + 2du_{q'}} < \sqrt{1 - d^2 + 2du_{q'}} = b_{q'} \tag{11}$$

and with Lemma 5.16, we know that all points covered by h' are contained in the ball with center c' and radius $b'_{q'}$. Because q' has maximum distance to h, either all points are covered by h' or none of the points is covered; the later happens if h' and the points in P' do not lie in one hemisphere. If all points in P' are covered by h', it follows that the ball with center c' and radius $b'_{q'} < b_{q'} \leq r$ encloses all points in P'. This is a contradiction to the assumption that the ball with center c and radius r is a smallest enclosing ball.

If h' does *not* cover the points in P', then we have to adapt our argumentation as follows. Instead of choosing h' to be perpendicular to c, we choose h' so that all points are covered; because the points in P' have to lie in one hemisphere, we know that there is such a h'. We choose c' to be the point on h' closest to the origin and by the assumption that all points of P' lie in one hemisphere, $d' = \|c'\|_2 = d - v > 0$. Because the original h' and the points in P' did not lie in one hemisphere, we know that $d - u_{q'} < 0$. This allows us to conclude that $v < u_q$ and plugging this into Equation 11 yields

$$b'_{q'} = \sqrt{1 - d^2 - v^2 + 2dv} < \sqrt{1 - d^2 - v^2 + 2du_{q'}} < \sqrt{1 - d^2 + 2du_{q'}} = b_{q'}.$$

Again, we can conclude that the ball with center c' and radius $b'_{q'} < b_{q'} \leq r$ encloses all points in P'; again a contradiction to the assumption that the ball with center c and radius r is the smallest enclosing ball for the point set P'. It follows that all points in P' are covered by h. □

Next, we prove that h is the $d-1$ dimensional hyperplane with maximum distance from the origin that is covering all points in P'.

Lemma 5.18. *For a point set P' lying on the unit sphere and being contained in one hemisphere, let the smallest enclosing ball have center c and radius r. The $d-1$ dimensional hyperplane h is perpendicular to the vector c and contains the point c. Then h is the $d-1$ dimensional hyperplane with maximum distance from the origin covering all points in P'.*

Proof. Assume that there is a $d-1$ dimensional hyperplane h' with larger distance from the origin than h and that h' is covering all points in P'. Let c' be the point on h' closest to the origin. Notice that c is contained in a facet of the convex hull of P'; let $Q' \subset P'$ be the points spanning this facet. Because all points in Q' have to be covered by h', c has as well to be covered by h'.

Consider the 2 dimensional plane spanned by the two vectors c and c'. Let us distinguish two cases and show for both of them that h' cannot have a larger distance to the origin than h has:

i. **h and h' have the same orientation:** If h and h' have both the same orientation and the distance between the origin and h' is larger than the distance between the origin and h, then c cannot be covered by h'; a contradiction.

ii. **h and h' have a different orientation:** Because h and h' have different orientations, the distance between c and c' has to be larger than zero. The point c has to be covered by h' and it follows with Pythagoras (see Figure 15) that $d' < d$; a contradiction to our assumption that h' is farther from the origin than h.

With Lemma 5.17, we can conclude that h is the $d-1$ dimensional plane with maximum distance from the origin that is covering P'. □

As we have seen before, a cone is defined by a $d-1$ dimensional sphere as base and by an apex; assuming — as it is guaranteed by our precondition — that all points in P' lie in one hemisphere. We choose the origin as apex, say that the basis is defined as the intersection of the d dimensional unit sphere with a $d-1$ dimensional hyperplane f, and assume that f intersects the unit sphere. We have to justify our choice by showing that

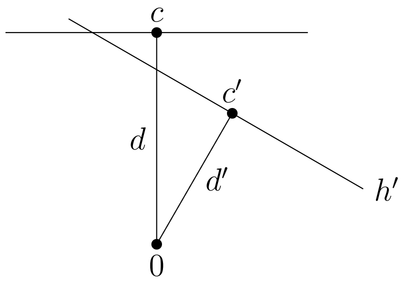


Figure 15: c' cannot be farther from the origin than c.

i. the intersection of the unit sphere and f produces a $d-1$ dimensional sphere

ii. every possible cone can be defined by a basis that is defined in this way.

For the first part, observe that we can rotate the space so that f is perpendicular to a coordinate axis; say x_1. Then the intersection between the unit sphere ($\sum x_i^2 = 1$) and f ($x_1 = const$) can be described as

$$\sum_{i=2}^{d} x_i^2 = 1 - x_1^2$$

and it follows immediately that the result of the intersection is a $d-1$ dimensional sphere with radius $r = \sqrt{1-x_1^2}$.

For the second part, notice that we can orient f arbitrarily and therefore the cone can have an arbitrary orientation. Further, notice that we can place f at arbitrary distance from the origin and therefore produce spheres with radius $0 \leq r < 1$. With the help of those spheres, we can produce cones with angles $\omega = \arctan\left(\frac{r}{d}\right)$; with $r = \sqrt{1-d^2}$, it follows that we can produce every $\omega \in [0, \frac{\pi}{2})$. This covers all cones we are interested in and it follows that we can produce arbitrary bases and therefore all cones we are interested in can be defined this way.

Next, we will show that the smallest enclosing cone problem can be reduced to the smallest enclosing ball problem. To do so, we will use the fact that h covers all points in P'.

Lemma 5.19. *Assume that for a point set P' lying on the unit sphere and being contained in one hemisphere, the smallest enclosing ball has center c and radius r. Let h be the $d-1$ dimensional hyperplane perpendicular to the vector c and containing the point c. Then the intersection of h and the unit sphere defines the base of the smallest enclosing cone.*

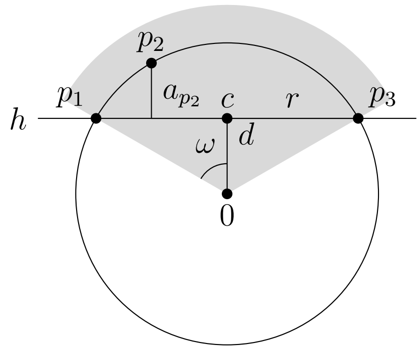


Figure 16: h covering P' defines a valid cone.

Proof. Assume that h has distance d from the origin. We are using the same notation as in Lemma 5.16 and proceed in two steps:

i. **h defines the basis of a valid enclosing cone:** Assume that the cone is defined by a $d-1$ dimensional sphere with center c and radius r as base. This definition can be translated to a cone with center vector c and angle $\omega = \arctan\left(\frac{r}{d}\right)$.

   Because h covers P', we know that for all points $p' \in P'$, we have $a_{p'} \geq 0$ and $t_{p'} \leq r$, and it follows that

$$\arctan\left(\frac{t_{p'}}{d + a_{p'}}\right) \leq \omega$$

   for all $p' \in P'$. This concludes the proof that h defines the base of a valid enclosing cone.

ii. **There is no smaller enclosing cone:** Let us assume that there *is* a smaller enclosing cone and show a contradiction. Let this cone be defined by h' and h' has a larger distance from the origin than h. Lemma 5.18 tells us that h' cannot cover all points in P'; let $q \in P'$ be a point that is not covered. Assume that h' has distance $d' > d$ from the origin, that c' is the center and $r' < r$ the radius of the basis. As we have seen above, we can equivalently define the cone by the center vector c' and the angle $\omega' = \arctan\left(\frac{r'}{d'}\right)$. Consider the 2 dimensional plane spanned by the two vectors c' and q and let us look at two cases. If the distance between q and the vector c' is larger than r', then it follows with Pythagoras (see Figure 17) that $\measuredangle c'0q > \omega'$; a contradiction. If the distance between q and the vector c' is smaller or equal to r', then the smallest enclosing cone and q are in different hemispheres; a contradiction.

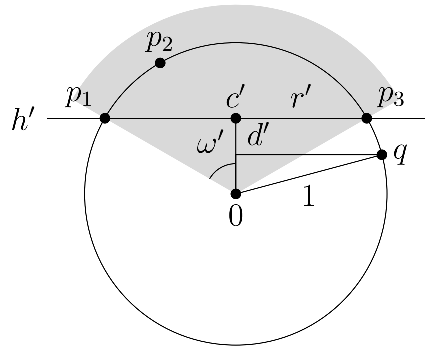


Figure 17: h' not covering P' cannot define a valid cone.

We can conclude that the intersection of h and the unit sphere defines the base of the smallest enclosing cone of the point set P'. □

With Lemma 5.19, we can give a simple procedure for the computation of the smallest enclosing cone using a reduction to the smallest enclosing ball problem:

i. Transform the point set P to P' by setting $p' = \frac{p}{\|p\|}$.

ii. Compute the smallest enclosing ball for P'; this returns a center c and a radius r.

iii. Center c defines the center vector of the cone and the angle is defined as

$$\omega = \arctan\left(\frac{r}{\|c\|}\right).$$

After having shown the equivalence of the smallest enclosing cone problem and the smallest enclosing ball problem with a geometric argumentation, we approach the problem from the optimization point of view.

Let us first start with the smallest enclosing ball problem. From Theorem 5.14, we know that the smallest enclosing ball problem can be defined as

$$\begin{array}{ll} \text{minimize} & x^T Q^T Q x - \sum_{j=1}^n x_j p_j^T p_j \\ \text{subject to} & \sum_{j=1}^n x_j = 1 \\ & x \geq 0 \end{array}$$

and for all points p_j lying on the unit sphere, we can rewrite this as

$$\begin{array}{ll} \text{minimize} & x^T Q^T Q x \\ \text{subject to} & \sum_{j=1}^n x_j = 1 \qquad (12) \\ & x \geq 0. \qquad (13) \end{array}$$

From [2] (p. 299), we know that

$$\begin{array}{rc} \text{minimize} & \frac{1}{2}x^T H x + d^T x \\ \text{subject to} & Ax \leq b \end{array}$$

is equivalent to

$$\begin{array}{rc} \text{maximize} & -\frac{1}{2}x^T H x - b^T u \\ \text{subject to} & Hx + A^T u = -d \\ & u \geq 0. \end{array}$$

Conditions 12 and 13 can be formulated in the form $Ax \leq b$ with

$$A_{1,i} = 1,\ A_{2,i} = -1,\ A_{i+2,i} = -1$$

for $i \in [d]$,

$$b_1 = 1,\ b_2 = -1,$$

and all other entries zero. Further, $d = 0$ holds and we can follow that

$$\begin{array}{rc} \text{minimize} & x^T Q^T Q x \\ \text{subject to} & \sum_{j=1}^{n} x_j = 1 \\ & x \geq 0. \end{array}$$

is equivalent to

$$\begin{array}{rc} \text{maximize} & -\frac{1}{2}x^T Q^T Q x - b^T u \\ \text{subject to} & Q^T Q x + A^T u = 0 \\ & u \geq 0. \end{array}$$

From Theorem 5.14, we know that $Qx = c$ and that the jth column of Q contains p_j. We can therefore rewrite the optimization problem as

$$\begin{array}{rc} \text{maximize} & -\frac{1}{2}c^T c - u_1 + u_2 \\ \text{subject to} & \forall p_j \in P' :\ p_j^T c + u_1 - u_2 - u_{2+j} = 0 \\ & u \geq 0. \end{array} \tag{14}$$

Notice that with $u \geq 0$, we can reformulate Condition 14 as

$$\forall p_j \in P' :\ p_j^T c + u_1 - u_2 = u_{2+j} \geq 0 \tag{15}$$

and we know that this can be interpreted as points lying on one side of a hyperplane h that is defined by c and $y = u_1 - u_2$. With Theorem 5.14, c lies in the convex hull of P'. This means that

$$c = \sum_{p_j \in P'} a_{p_j} p_j,\quad \sum_{p_j \in P'} a_{p_j} = 1,\ \ \forall p_j \in P' : a_{p_j} \geq 0$$

and therefore with Condition 15

$$\sum_{p_j \in P'} a_{p_j} \left(p_j^T c + u_1 - u_2\right) = \left(\sum_{p_j \in P'} a_{p_j} p_j\right)^T c + \sum_{p_j \in P'} a_{p_j} (u_1 - u_2)$$
$$= c^T c + u_1 - u_2 \geq 0$$

holds. We can add the derived condition to the optimization problem without changing its solution and get

$$\begin{aligned}
\text{maximize} \quad & -\tfrac{1}{2} c^T c - u_1 + u_2 \\
\text{subject to} \quad & \forall p_j \in P' : p_j^T c + u_1 - u_2 \geq 0 && (16)\\
& c^T c + u_1 - u_2 \geq 0 && (17)\\
& u \geq 0.
\end{aligned}$$

Conditions 16 and 17 can be interpreted as points lying on one side of a hyperplane h defined by c and $y = u_1 - u_2$. From Condition 17, it follows that $y \geq -c^T c$ and we would like to claim that Condition 16 is fulfilled for $y = -c^T c$. Observe that the distance between the hyperplane h and the origin is given as

$$\frac{|-y|}{\|c\|_2}$$

and we can conclude that for $y = -c^T c$, the distance is $\|c\|_2$. As c has distance $\|c\|_2$ from the origin and as h is orthogonal to c (with [15]), it follows that c is contained in the hyperplane h for $y = -c^T c$. c is the point on the convex hull of P' that is closest to the origin and it follows that all points $p_j \in P'$ have to fulfill

$$p_j^T c - c^T c \geq 0$$

as otherwise, we could find a c' that is closer to the origin. It follows that Conditions 16 and 17 are fulfilled for $y = u_1 - u_2 = -c^T c$.

Notice that for $y \leq -c^T c$, the point c is not covered by the hyperplane defined by c and y and it follows with the fact that c is contained in the convex hull of P' that a point in P' is not covered and therefore Condition 16 is not fulfilled.

With Condition 17, we know that the objective function is upper bounded by

$$-\frac{1}{2} c^T c - y \leq -\frac{1}{2} c^T c + c^T c = \frac{1}{2} c^T c.$$

In order to satisfy $u \geq 0$, we set $u_1 = 0$ and $u_2 = -y = c^T c$ and as for a given point set P', c is uniquely determined, it follows that the problem is maximized for $y = u_1 - u_2 = -c^T c$ with solution $\frac{1}{2} c^T c$.

After having analyzed the smallest enclosing ball problem, let us have a look at the smallest enclosing cone problem. The smallest enclosing cone problem can be defined as

$$\begin{array}{rc} \text{minimize} & v \\ \text{subject to} & \forall p_j \in P' : \arccos\left(\frac{p_j^T c}{\|p_j\|\|c\|}\right) \leq v. \end{array}$$

As we have seen before, the function $\arccos(t)$ is a strictly decreasing function in the interval we are considering and therefore we can rewrite the optimization problem as

$$\begin{array}{rc} \text{maximize} & v \\ \text{subject to} & \forall p_j \in P' : \frac{p_j^T c}{\|p_j\|\|c\|} \geq v. \end{array} \tag{18}$$

Condition 18 can be reformulated as

$$\forall p_j \in P' : p_j^T c - v\|c\|\|p_j\| \geq 0$$

whereas $\|p_j\| = 1$ and we write

$$\begin{array}{rc} \text{maximize} & v \\ \text{subject to} & \forall p_j \in P' : p_j^T c - v\sqrt{c^T c} \geq 0. \end{array} \tag{19}$$

Notice that Conditions 16 and 19 have the same form. By the reasoning that we have made for Condition 16, we know that Condition 19 is fulfilled for $v = \sqrt{c^T c}$ and that there is no $v \geq \sqrt{c^T c}$ so that Condition 19 is fulfilled. It follows that the two optimization problems fulfill in the optimum the same conditions and therefore the two problems are equivalent.

On a side note, observe that c and $v\sqrt{c^T c}$ define a hyperplane and according to [15], the distance between this hyperplane and the origin is given as

$$\frac{v\sqrt{c^T c}}{\sqrt{c^T c}} = v.$$

We can conclude that the smallest enclosing cone problem is equivalent to finding the hyperplane with maximum distance from the origin that — as stated by Condition 19 — covers all points in P'.

This concludes our alternative proof showing that the smallest enclosing ball problem and the smallest enclosing cone problem are the same.

6 Non-SC Point Sets and the Miniball

We present miniball algorithms for two types of point sets that are not part of the family of SC point sets. The two algorithms are more or less obvious but we state them here for completeness.

6.1 Points with L_∞ Distance Function

In L_∞, the distance function $d_p(c)$ can be written as

$$d_p(c) = \sqrt[\infty]{\sum |p_i - c_i|^\infty} = \max_i \{|p_i - c_i|\}.$$

Notice that $d_p(c)$ has level sets that are *not* strictly convex but we still investigate the smallest enclosing ball problem. Using the inequality

$$\max_i\{|p_i - c_i|\} \le r$$

for the miniball problem, the ball can be interpreted as a hypercube; the problem of finding the smallest ball is reduced to finding the smallest hypercube that contains all points.

The minimum hypercube can be found in time $O(d|P|)$ by going through all points $p \in P$ and computing the smallest and the largest coordinate in each dimension; let us denote those coordinates by l_i and u_i for the smallest resp. largest coordinate in dimension i. The side length b of the smallest hypercube containing all points $p \in P$ is $b = \max_{i\in[d]}\{u_i - l_i\}$. Notice that the coordinates of the center c are fixed only for dimensions i with $u_i - l_i = b$. For all other dimensions,

$$\frac{u_i + l_i}{2} - (b - (u_i - l_i)) \le c_i \le \frac{u_i + l_i}{2} + (b - (u_i - l_i)).$$

The given algorithm does not return a unique smallest enclosing ball. It is important to realize that this is not a problem of the algorithm at hand but rather a property of the miniball in L_∞.

Notice that we can extend this algorithm to the L_∞ anisotropic setting. In the anisotropic setting for L_∞, the distance function is defined as

$$d_p(c) = \sqrt[\infty]{\sum \left|\frac{p_i - c_i}{s_i}\right|^\infty} = \max_i \left\{\frac{|p_i - c_i|}{s_i}\right\}$$

and $b = \max_i\{\frac{u_i - l_i}{s_i}\}$. Extending to the anisotropic setting is possible because the *axes of the coordinate systems* are considered separately.

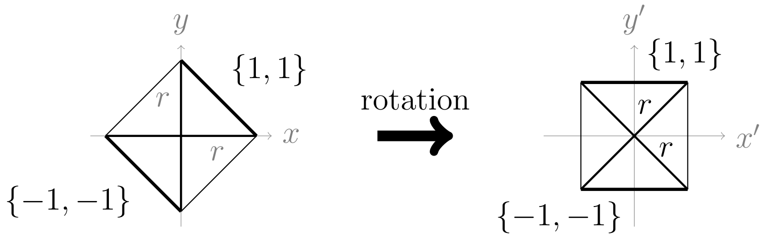


Figure 18: The cross polytope in 2 dimensions. The two faces $\{1, 1\}$ and $\{-1, -1\}$ are opposite and by applying an appropriate transformation matrix, we can transform the space so that the two faces $\{1, 1\}$ and $\{-1, -1\}$ are parallel to the x' axis.

6.2 Points with L_1 Distance Functions

As for L_∞ distance functions, L_1 distance functions have *not* strictly convex level sets but again, we investigate the miniball problem. The smallest enclosing ball for L_1 is a cross polytope with 2^d faces. For sake of simplicity, assume that the center of the cross polytope is located at the origin and that the cross polytope has radius r. The faces are determined by d vectors of length r that are aligned with the coordinate axes. Two opposite faces — say the two faces defined by vectors in positive direction for all coordinate axes and by vectors in negative direction for all coordinate axes — have the same orientation.

To find the smallest enclosing ball in L_1, we can employ the same ideas as given for the smallest enclosing ball in L_∞. For every two opposite faces, we can find a rotation matrix so that the two faces are perpendicular to one coordinate axis; in Figure 18, the two faces are perpendicular to the y' axis. Finding the smallest and the largest coordinate along this axis gives us the minimum distance between the two faces; let us denote the smallest coordinate by l_i and the largest by u_i. Notice that for every pair of opposite faces, we have to employ a separate rotation.

The smallest cross polytope containing all points has radius $r = \max_i\{\frac{u_i - l_i}{2}\}$. To compute this radius, we have to employ 2^{d-1} rotations and every rotation costs $d^2|P|$.

7 Approximation and ϵ-Core Sets

In the previous section, we have shown that the smallest enclosing ball of an SC point set can be computed in expected time linearly dependent on the number of points but unfortunately exponentially dependent on the dimension. In this section, we try to get rid of this exponential dependence of the dimension and investigate approximative solutions. An approximative solution for the miniball problem is a ball that is *not much larger* than the exact miniball and that contains all points of the point set.

Definition 7.1. *Let P be a point set with the smallest enclosing ball $MB(P)$ having center c and radius r and let $AMB(P)$ be a ball with center c_A and radius r_A so that all points in P are enclosed in $AMB(P)$. We say that $AMB(P)$ is an $(1+\epsilon)$-approximation of $MB(P)$ if $r_A \leq (1+\epsilon)r$.*

Bădoiu and Clarkson introduced in [4, 5] the concept of ϵ-core sets. First, let us define ϵ-core sets.

Definition 7.2. *Let P be a point set, $S \subset P$, and say that the smallest enclosing ball $MB(S)$ has center c_S and radius r_S. The point set S is an ϵ-core set if the ball with center c_S and radius $(1+\epsilon)r_S$ contains all points of P and if $|S| = f(\epsilon)$ for some function f that is independent on $|P|$ and d.*

Observe that the smallest enclosing ball of an ϵ-core set is a $(1+\epsilon)$-approximation. With $S \subset P$, $r_S \leq r$ has to hold and we know that all points in P are contained in a ball with center c_S and radius $(1+\epsilon)r_S \leq (1+\epsilon)r$. It follows that whenever we have an ϵ-core set, we can conclude that there is a $(1+\epsilon)$-approximation. Notice though that the opposite direction does *not* hold as the non-existence of ϵ-core sets does not imply that there cannot be a $(1+\epsilon)$-approximation.

ϵ-core sets are interesting because the size of the ϵ-core set S is dependent only on ϵ and neither on the dimension nor on the size of the point set. Assume that $P \subset \mathbb{R}^d$ and therefore $S \subset \mathbb{R}^d$. With $|S| = f(\epsilon)$, we can reduce the point set S to $\lceil f(\epsilon) \rceil$ dimensional space. In this reduced space, Algorithm 2 can be applied and the miniball can be computed. If $f(\epsilon) < d$ holds, then we can conclude that the computation of $MB(S)$ is not exponentially dependent on d but rather exponentially dependent on $\lceil f(\epsilon) \rceil$.

Let us give a short outline for this section. First, we analyze point sets of points with L_2 distance functions. In [4, 5], Bădoiu and Clarkson show that ϵ-core sets exist for such point sets by presenting algorithms. In Section 7.1, we will show that removing an *optimal* point from a point set can decrease the radius by at most a constant factor whereas a point P is optimal if

$MB(P \setminus \{p\})$ is as large as possible. We are interested in a bound on the decrease for the radius as this seems to present a simple core set algorithm. *Always remove the point so that the remaining points have a miniball that is as large as possible.* Even though we are able to show a constant upper bound for the decrease in radius, we are *not* able to prove the existence of core sets with that knowledge though we suspect that this is due to our analysis and not due to the nature of the problem.

In Section 7.2, we present an example showing that for SC point sets, there are configurations that cannot have ϵ-core sets. In Section 7.3, we show that the decrease in radius for SC point sets is not bounded by a constant factor but rather by a factor that is dependent on the distance functions involved. We present this result without application but assume that there is a connection between this result and the existence of core sets for SC point sets with a size that is dependent on ϵ and the distance functions. In Section 7.4, we present two core set algorithms and show for the first that it is not working for SC point sets and conjecture that the second algorithm produces core sets with a size dependent on ϵ and the distance functions.

7.1 Analysis of Point Sets with L_2 Distance Functions

The concept of ϵ-core sets is intuitively coupled with the property that removing an *optimal point* from the basis does not produce a ball with *much smaller* radius. With optimal point, we mean a point q from the point set P so that removing q results in a point set with a smallest enclosing ball that is as large as possible. First, we analyze the situation for the distance function L_2 and show that the decrease in radius is bounded. Second, we will use this fact to give hints at how to prove the existence of ϵ-core sets in an alternative way though pieces and bits are missing.

First, let us present a Lemma that will be employed in later parts of this section.

Lemma 7.3. *For a point set $P \subset \mathbb{R}^d$ with $\|p - q\| = u$ for all $p, q \in P$ and with $|P| = d + 1$, the smallest enclosing ball $MB(P)$ has center*

$$c_d = \frac{1}{d+1} \sum_{p \in P} p$$

and radius

$$r_d = \sqrt{\frac{d}{2(d+1)}} u.$$

Proof. Say that a point set P with $\|p-q\| = u$ for all $p, q \in P$ is a point set in *equilateral position* and notice that a point set in equilateral position is a regular d-simplex. With Theorem 3.2 from [10], we know that the centroid and the circumcenter of the regular d-simplex coincide and we can therefore conclude that the circumcenter is contained in the convex hull of P. With Lemma 5.2, the circumball and the smallest enclosing ball of the d-simplex have to be equal and it follows that the distance between the centroid and an arbitrary $p \in P$ is equal to the radius of the smallest enclosing ball. From [17], we know that the distance between the centroid and a $p \in P$ is

$$r_d = \sqrt{\frac{d}{2(d+1)}}u$$

and our claim follows. □

To start our analysis of the maximal decrease in radius, we show that a point set in equilateral position results in the worst possible decrease in radius for an optimal point.

Lemma 7.4. *For a point set $P \subset \mathbb{R}^d$ with $|P| \geq d+1$, say that $MB(P)$ has radius r_d and center c_d and that for all $p \in P$, $MB(P \setminus \{p\})$ has radius $r_{d-1,p}$ and center $c_{d-1,p}$. Then*

$$\min_{p \in P} \frac{r_d}{r_{d-1,p}}$$

is maximized if $|P| = d+1$ and P is in equilateral configuration.

Proof. We proceed with a case distinction:

1. $|P| > d+1$: With Lemma 3.4, there is an $S \subset P$ with $|S| \leq d+1$ so that $MB(S) = MB(P)$. Let $p \in S$ be an optimal point and observe that with $S \subset P$, the radius of $MB(S \setminus \{p\})$ has to be smaller or equal compare to the radius of $MB(P \setminus \{p\})$. It follows that the decrease in radius for S is at least as large as for P and we can assume that $|P| \leq d+1$.

2. $|P| = d+1$: We may assume that all points $p \in P$ lie on the boundary of $MB(P)$ as otherwise removing a point that is not lying on the boundary results in $r_{d-1,p} = r_d$ and this can clearly not maximize $\min_{p \in P} \frac{r_d}{r_{d-1,p}}$.

   With Theorem 5.14, we know that $c_{d-1,p}$ is the point in the convex hull $conv(P \setminus \{p\})$ that is closest to c_d. With the observation that the points

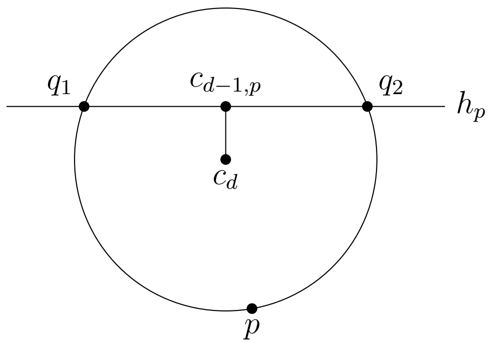


Figure 19: Points q_1 and q_2 that lie on the hyperplane h_p fulfill $(q_i - c_{d-1,p})^T(c_{d-1,p} - c_d) = 0$.

in $P \setminus \{p\}$ lie on a $(d-1)$-dimensional hyperplane h_p, it follows (see Figure 19) that for all $q \in P \setminus \{p\}$,

$$(q - c_{d-1,p})^T(c_{d-1,p} - c_d) = 0. \tag{20}$$

From Lemma 7.3, we know that for a point set in equilateral position,

$$r_d = \sqrt{\frac{d}{2(d+1)}}u$$

holds for u being the distance between any two points of P. Let $e_{d,p} = \|c_{d-1,p} - c_d\|$. Because of Equation 20, we know with Pythagoras that

$$e_{d,p} = \sqrt{r_d^2 - r_{d-1,p}^2}.$$

For a point set P in equilateral position, it follows that

$$\begin{aligned} e_{d,p} =& \sqrt{r_d^2 - r_{d-1,p}^2} \\ =& \sqrt{\frac{d}{2(d+1)}u^2 - \frac{d-1}{2d}u^2} \\ =& \frac{u}{\sqrt{2d(d+1)}} \\ =& \frac{r_d}{d}. \end{aligned}$$

Let us restate Lemma 5.1 from [4] and give a short sketch of their proof:

Lemma 7.5. ***(Lemma 5.1 from [4])*** *Let B' be the largest ball contained in a simplex T, such that B' has the same center as the minimum enclosing ball $B(T)$. Then*

$$r_{B'} \leq \frac{r_{B(T)}}{d}.$$

Proof. Observe first that

$$\left(\frac{r_{B'}}{r_{B(T)}}\right)^d = \frac{vol(B')}{vol(B(T))}.$$

Say that $e(T)$ is the ellipsoid with maximum volume inside T and that $E(T)$ is the ellipsoid with minimum volume enclosing T. $vol(B') \leq vol(e(T))$ and $vol(B(T)) \geq vol(E(T))$ hold and therefore

$$\left(\frac{r_{B'}}{r_{B(T)}}\right)^d \leq \frac{vol(e(T))}{vol(E(T))}.$$

Affine mappings preserve volume ratios [19] and we can therefore assume that T is the regular d-simplex.

For a regular d-simplex T, the smallest enclosing ellipsoid is a ball. Assume not and notice that we can define a mapping for the vertices of the regular d-simplex and it follows that there has to be more than one smallest enclosing ellipsoid. But from [37], we know that the smallest enclosing ellipsoid is unique and it follows that the smallest enclosing ellipsoid is a ball.

Further, the largest ellipsoid enclosed in a regular d-simplex T is a ball. Assume not and notice that we can again define a mapping for the vertices of the regular d-simplex and we can conclude that there are at least two largest enclosed ellipsoids. By Theorem 3 from [19], the ellipsoid of maximum volume that is contained in a compact body is unique. It follows that the ellipsoid with maximum volume contained in T is a ball.

Because both the largest enclosed and the smallest enclosing ellipsoids are balls, we can conclude that the centroid of the regular d-simplex is the center of both balls. For those two balls, we can apply the *John Ellipsoid Theorem* and it follows from [19, 20] that

$$\frac{r_{e(T)}}{r_{E(T)}} = \frac{1}{d}.$$

As both $e(T)$ and $E(T)$ are balls, we can conclude that

$$\left(\frac{r_{B'}}{r_{B(T)}}\right)^d \leq \frac{vol(e(T))}{vol(E(T))} = \left(\frac{r_{e(T)}}{r_{E(T)}}\right)^d = \left(\frac{1}{d}\right)^d$$

and

$$\frac{r_{B'}}{r_{B(T)}} \leq \frac{1}{d}$$

follows. □

Say that the largest ball with center c_d that is contained in $conv(P)$ has radius $r_{B'}$. Because $c_{d-1,p}$ is the point on $conv(P \setminus \{p\})$ that is closest to c_d, it follows that $r_{B'} = \min_{p \in P} e_{d,p}$. With Lemma 7.5, we can conclude that

$$\min_{p \in P} e_{d,p} = r_{B'} \leq \frac{r_d}{d}. \tag{21}$$

Observe that for a point set P in equilateral position, $e_{d,p} = \frac{r_d}{d}$ for all $p \in P$ and Equation 21 holds with equality.

As $r_{d-1,p} = \sqrt{r_d^2 - e_{d,p}^2}$, r_d is fixed, and $e_d = \min\{e_{d,p}\}$ is maximal for a point set in equilateral position, it follows that $\max_{p \in P} r_{d-1,p}$ is minimal for a point set in equilateral position. Further, this allows us to conclude that

$$\min_{p \in P} \frac{r_d}{r_{d-1,p}}$$

is maximized for a point set in equilateral configuration.

Summarizing the two cases, we can conclude that a point set P in equilateral configuration maximizes

$$\min_{p \in P} \frac{r_d}{r_{d-1,p}}.$$

□

Notice that Lemma 7.4 can as well be used for point sets $P \subset \mathbb{R}^d$ with $|P| < d+1$ as such a point set can be mapped to $\mathbb{R}^{|P|-1}$ and then fulfills the precondition of the Lemma.

After having shown that point sets in equilateral position are the worst case for decrease in radius, we show that the decrease for this worst case is bounded from above by a constant factor.

Lemma 7.6. *For a point set $P \subset \mathbb{R}^d$ with $|P| = d+1$, assume that $MB(P)$ has radius r_d and for $p \in P$, $MB(P \setminus \{p\})$ has radius $r_{d-1,p}$.*

$$\min_{p \in P} \frac{r_d}{r_{d-1,p}} \leq \frac{1}{\cos\left(\frac{\pi}{6}\right)}$$

holds for all $d \geq 2$.

Proof. We proceed by induction and first show the base case for $P \subset \mathbb{R}^2$. From Lemma 7.4, we know that the point set P with the largest value for

$$\min_{p \in P} \frac{r_d}{r_{d-1,p}}$$

has to be in equilateral position. With simple trigonometric reasoning, we get $r_{1,p} = r_2 \cos\left(\frac{\pi}{6}\right)$ and therefore

$$\frac{r_2}{r_{1,p}} = \frac{r_2}{r_2 \cos\left(\frac{\pi}{6}\right)} = \frac{1}{\cos\left(\frac{\pi}{6}\right)} \approx 1.1547 > 1.$$

This already concludes the analysis for the base case. Next, we look at $P \subset \mathbb{R}^d$. To bound the ratio

$$\frac{r_d}{r_{d-1,p}},$$

we show that

$$\frac{r_d}{r_{d-1,p}} \leq \frac{r_{d-1}}{r_{d-2,p}} \tag{22}$$

holds. Observe that the point set has to be in equilateral position and it is therefore a regular d-simplex. With Lemma 7.3, we know that that

$$r_d = \sqrt{\frac{d}{2(d+1)}}u$$

whereas u stands for the edge length of the d-simplex. Notice that removing a vertex from a regular d-simplex produces a regular $(d-1)$-simplex and

$$r_{d-1,p} = r_{d-1} = \sqrt{\frac{d-1}{2d}}u$$

follows. The definitions for r_d and $r_{d-1,p}$ yield

$$f(d) = \frac{r_d}{r_{d-1,p}} = \frac{d}{\sqrt{(d+1)(d-1)}}.$$

Differentiating $f(d)$ results in

$$\frac{\partial}{\partial d}\left(\frac{d}{\sqrt{(d+1)(d-1)}}\right) = \frac{-1}{((d+1)(d-1))^{\frac{3}{2}}}$$

and it follows that $f(d)$ is a strictly decreasing function for all $d \geq 2$. We can conclude that

$$\frac{r_d}{r_{d-1,p}} \leq \frac{r_{d-1}}{r_{d-2,p}}$$

holds and therefore

$$\frac{r_d}{r_{d-1,p}} \leq \frac{1}{\cos\left(\frac{\pi}{6}\right)}$$

for all $d \geq 2$. With Lemma 7.4 it follows that for any point set,

$$\min_{p \in P} \frac{r_d}{r_{d-1,p}} \leq \frac{1}{\cos\left(\frac{\pi}{6}\right)}$$

holds. □

Based on the knowledge that $\min_{p \in P} \frac{r_d}{r_{d-1,p}}$ is bounded from above by a constant factor, it seems reasonable to assume that we can use a similar technique to show that there are core sets of size $f(\epsilon)$ for some function f that is independent on $|P|$ and d.

First, let us show that the distance between the centers of two smallest enclosing balls $MB(P)$ and $MB(P \setminus \{p\})$ is maximal for a point set in equilateral configuration.

Lemma 7.7. *Given a point set $P \subset \mathbb{R}^d$ with $p \in P$ maximizing $r_{d-1,p}$. Let $MB(P)$ have center c_d and radius r_d and $MB(P \setminus \{p\})$ center $c_{d-1,p}$ and radius $r_{d-1,p}$. Say that $e_d = \|c_d - c_{d-1,p}\|$. For a fixed r_d, e_d is maximal for a point set P in equilateral position with $|P| = d+1$.*

Proof. This follows directly from the proof of Lemma 7.4. Recall that we have shown in that proof that e_d is maximal for a point set P in equilateral position with $|P| = d+1$. □

Based on Lemma 7.7, we show that e_d is bounded from above by a constant factor.

Lemma 7.8. *For a point set $P \subset \mathbb{R}^d$ and $p \in P$ maximizing $r_{d-1,p}$, the distance e_d between the centers of $MB(P)$ and $MB(P \setminus \{p\})$ is bounded from above by*

$$e_d \leq \frac{r_d}{\sqrt{6}}.$$

Proof. With Lemma 7.7, we know that e_d is maximal if the point set P is in equilateral position. For such a point set P, we know with Lemma 7.3 that

$$r_d = \sqrt{\frac{d}{2(d+1)}}u \tag{23}$$

where u is the distance between any two points in the equilateral point set P. As we have argued in the proof of Lemma 7.4, $e_d = \sqrt{r_d^2 - r_{d-1}^2}$ and plugging the definition of r_d into this yields

$$e_d = \sqrt{r_d^2 - r_{d-1}^2} = \frac{u}{\sqrt{2d(d+1)}} = \frac{r_d}{d}.$$

Differentiating e_d produces

$$\frac{\partial}{\partial d}\frac{u}{\sqrt{2d(d+1)}} = -\frac{(2d+1)u}{(2d(d+1))^{\frac{3}{2}}} < 0$$

and the last inequality holds for all $d \geq 1$ and $u > 0$. It follows that e_d is strictly decreasing and that

$$e_d \leq e_2 = \frac{u}{2\sqrt{3}} \tag{24}$$

for all $d \geq 2$. Reformulating Equation 23 gives

$$u = \sqrt{\frac{2(d+1)}{d}}r_d \tag{25}$$

and differentiating Equation 25 yields

$$\frac{\partial}{\partial d}\sqrt{\frac{2(d+1)}{d}}r_d = -\frac{1}{\sqrt{2(d+1)}d^{\frac{3}{2}}}r_d \tag{26}$$

which is negative for all $d \geq 1$. It follows that u is strictly decreasing as d is increasing and assuming that r_d is fixed.

$$u \leq \sqrt{2}r_d$$

follows and plugging this into Equation 24 yields

$$e_d \leq \frac{r_d}{\sqrt{6}}.$$

□

One is tempted to use Lemma 7.8 to bound the core set size. We know that for all points $p \in P$, $\|p - c_d\| \le r_d$. In the worst case, p, c_d, and c_{d-1} are on a common line and $\|p - c_{d-1}\| \le r_d + e_d$ follows. More generally, $\|p - c_{d-l}\| \le r_d + \sum_{i=d-l}^{d} e_i$ holds. The definition of the core set states that

$$\forall p \in P : \|p - c_{d-l}\| \le (1 + \epsilon) r_d$$

is fulfilled with core set size $f = d - l$. It follows that an f fulfilling

$$r_d + \sum_{i=d-l}^{d} e_i \le (1 + \epsilon) r_d$$

is the size of a possible core set. Plugging in the upper bound from Lemma 7.8 results in

$$\begin{aligned} r_d + \sum_{i=d-l}^{d} e_i \le & r_d + l \frac{r_d}{\sqrt{6}} \\ = & \left(1 + \frac{l}{\sqrt{6}}\right) r_d \\ \le & (1 + \epsilon) r_d \end{aligned}$$

Reformulating this yields

$$l \le \sqrt{6}\epsilon.$$

As the core set size is $f = d - l$,

$$f \ge d - \sqrt{6}\epsilon$$

and it follows that we are not able to establish a bound only dependent on ϵ in this way.

Our analysis has two main problems. First, our upper bound for e_d is only tight for $d = 2$ and $e_d < \frac{r_d}{\sqrt{6}}$ holds for $d > 2$. This could be improved by bounding $\sum_{i=d-l}^{d} e_i$ directly instead of bounding $e_d, e_{d-1}, ...$ separately and then taking the sum of the bounds. Second, the points p, c_d, c_{d-1}, and c_{d-2} lie not on a common line and therefore summing up the e_i's gives us only a rough upper bound. Sure, it should be possible to sum up e_i's considering the position of the c_{d-l+i}'s but it is no immediately obvious that a point set in equilateral position is really the worst case for the distance $\|p - c_{d-l+i}\|$. So far, we have only shown that e_d is maximal in the equilateral case. It could be that the distance $\|p - c_{d-2}\|$ is larger for another point set that has a more obtuse angle $\sphericalangle p c_{d-1} c_{d-2}$ than a point set in equilateral position.

Next we define a better upper bound for $\sum_{i=d-l}^{d} e_i$.

Lemma 7.9. *For any point set $P \subset \mathbb{R}^d$ and $p \in P$ maximizing $r_{d-1,p}$, let e_d be the distance between the centers of $MB(P)$ and $MB(P \setminus \{p\})$.*

$$\sum_{i=d-l}^{d} e_i < r_d \sqrt{\frac{d+1}{d}} \left(\frac{-l}{2d(d-l)} + \ln(d) - \ln(d-l-1) \right)$$

holds.

Proof. We proceed in two steps. First, we put r_d and r_{d-i} for a point set in equilateral position into relation. Second, we use this relation to derive an upper bound on $\sum_{i=d-l}^{d} e_i$.

With Lemma 7.7, we know that e_d is maximal for a point set in equilateral position. From Lemma 7.3, we know that for a point set in equilateral position,

$$r_d = \sqrt{\frac{d}{2(d+1)}} u$$

holds and

$$r_{d-1} = \frac{\sqrt{(d+1)(d-1)}}{d} r_d$$

follows. For r_{d-2}, we have

$$r_{d-2} = \frac{\sqrt{(d-2)(d-1)d(d+1)}}{(d-1)d} r_d$$

and more generally

$$\begin{aligned} r_{d-i} =& \frac{\sqrt{\prod_{j=d-i}^{d-1} j \prod_{j=d-i}^{d-1} (j+2)}}{\prod_{j=d-i}^{d-1} (j+1)} r_d \\ =& \frac{\sqrt{\prod_{j=d-i}^{d-1} j \prod_{j=d-i+2}^{d+1} j}}{\prod_{j=d-i+1}^{d} j} r_d \\ =& \frac{\sqrt{(d-i)(d-i+1)d(d+1)} \prod_{j=d-i+2}^{d-1} j}{\prod_{j=d-i+1}^{d} j} r_d \\ =& \sqrt{\frac{(d-i)(d+1)}{(d-i+1)d}} r_d. \end{aligned} \tag{27}$$

Using Equation 27, we try to find a good upper bound for $\sum_{i=d-l}^{d} e_i$. In the proof of Lemma 7.8, we have seen that $e_i = \frac{r_i}{i}$ for a point set in equilateral position. With Equation 27,

$$\begin{aligned}e_{d-i} =& \frac{r_{d-i}}{d-i} \\ =& \sqrt{\frac{d+1}{d(d-i)(d-i+1)}} r_d.\end{aligned}$$

Plugging this into the summation yields

$$\sum_{i=d-l}^{d} e_i = r_d \sqrt{\frac{d+1}{d}} \sum_{i=0}^{l} \frac{1}{\sqrt{(d-i)(d-i+1)}}$$

and it remains to find an explicit formula for

$$\sum_{i=0}^{l} \frac{1}{\sqrt{(d-i)(d-i+1)}}.$$

Observe that

$$\sqrt{(d-i)(d-i+1)} > \sqrt{(d-i)(d-i)} = d-i$$

and therefore

$$\frac{1}{\sqrt{(d-i)(d-i+1)}} < \frac{1}{d-i}$$

holds and this bound is essentially tight. Plugging this upper bound back into the summation gives us

$$\sum_{i=0}^{l} \frac{1}{\sqrt{(d-i)(d-i+1)}} < \sum_{i=0}^{l} \frac{1}{d-i}$$

whereas the right side of the equation is related to the harmonic number. For the harmonic number, the following bounds are known:

$$\frac{1}{2(n+1)} + \ln(n) + \gamma < \sum_{i=1}^{n} \frac{1}{i} < \frac{1}{2n} + \ln(n) + \gamma.$$

Further,

$$\begin{aligned}\sum_{i=0}^{l} \frac{1}{d-i} =& \sum_{i=1}^{d} \frac{1}{i} - \sum_{i=1}^{d-l-1} \frac{1}{i} \\ <& \frac{1}{2d} + \ln(d) + \gamma - \frac{1}{2(d-l)} - \ln(d-l-1) - \gamma \\ =& \frac{-l}{2d(d-l)} + \ln(d) - \ln(d-l-1)\end{aligned}$$

and using this in the summation yields

$$\sum_{i=d-l}^{d} e_i < r_d\sqrt{\frac{d+1}{d}}\left(\frac{-l}{2d(d-l)} + \ln(d) - \ln(d-l-1)\right)$$

and our claim follows □

Again, one is tempted to use Lemma 7.9 to show that ϵ-core sets exist. As we have seen,

$$\|p - c_{d-l}\| \leq r_d + \sum_{i=d-l}^{d} e_i$$

and therefore showing that

$$r_d + \sum_{i=d-l}^{d} e_i \leq (1+\epsilon)r_d$$

would imply

$$\|p - c_{d-l}\| \leq (1+\epsilon)r_d.$$

Using Lemma 7.9, we get

$$\begin{aligned} r_d + \sum_{i=d-l}^{d} e_i <& r_d\left(1 + \sqrt{\frac{d+1}{d}}\left(\frac{-l}{2d(d-l)} + \ln(d) - \ln(d-l-1)\right)\right) \\ \leq& (1+\epsilon)r_d \end{aligned}$$

and

$$\sqrt{\frac{d+1}{d}}\left(\frac{-l}{2d(d-l)} + \ln(d) - \ln(d-l-1)\right) \leq \epsilon$$

follows. From this equation, it is obvious that the core set size $f = d - l$ cannot be expressed solely using ϵ and it follows that this analysis does not yield an ϵ-core set as it is known to exist.

We can conclude that our analysis is still not good enough and we identify two sources where we loose precision. First, the bound given in Lemma 7.9 for $\sum_{i=d-l}^{d} e_i$ is not entirely tight. The problem in the proof is that the bound for

$$\sum_{i=0}^{d} \frac{1}{\sqrt{i(i+1)}}$$

is not tight though notice that the error made by

$$\sum_{i=0}^{d} \frac{1}{\sqrt{i(i+1)}} < \sum_{i=0}^{d} \frac{1}{i}$$

is *really* small and we are not making a significant mistake here.

Second, we still ignore the fact that the points p, c_d, c_{d-1}, and c_{d-2} do not lie on one line. As we have mentioned before, it is difficult to argue that a point set in equilateral position is indeed the worst possible configuration. As a consequence, we do not proceed.

7.2 Arbitrary Decrease in Radius and no ϵ-Core Sets for SC Point Sets

In this section, we consider SC point sets and will show that removing an optimal point can decrease the radius of the smallest enclosing ball arbitrarily. As we will see, this implies that there cannot be ϵ-core sets with size independent on the involved distance functions.

Notice that we do *not* consider $d = 1$ as removing one of the two points defining a smallest enclosing ball gives us always a smallest enclosing ball with radius 0.

To show that the decrease in radius can be arbitrarily large, let us construct an example. First, we define the position of the points and second, we construct anisotropic distance functions for all points from the point set. We will show that for this point set, the decrease in radius as it was defined in the previous section is arbitrarily large.

Assume that we are given a d-simplex defined by a vertex at the origin $\mathbf{0}$ and vertices at e_i for $i \in [d]$ and e_i being the unit vector along the i'th dimension. Observe that this d-simplex is defined by $d+1$ $(d-1)$-simplices; let us call those simplices *faces*. A face is defined by d vertices and we use this fact to name the faces; face i does *not* contain e_i. One of the $d+1$ faces contains all e_i for $i \in [d]$ but not $\mathbf{0}$; this is face $d+1$.

Based on the faces, we define the point set P. Assume that the largest sphere $IS(c_d, r'_d)$ contained in the d-simplex and touching all faces has center c_d and radius r'_d. By symmetry, we know that $c_d = \lambda I$ for some $\lambda > 0$ and $I = \sum_{i=1}^{d} e_i$. For face i, say that p_i is the unique point contained in both face i and the sphere $IS(c_d, r'_d)$. By the fact that $IS(c_d, r'_d)$ touches the faces in exactly one point, we know that the line containing p_i and c_d is perpendicular to the face i and it follows that $p_i = \sum_{j \in [d] \setminus \{i\}} \lambda e_j$. As the distances $\|p_i - c_d\|$ for $i \in [d]$ and $\|p_{d+1} - c_d\|$ have to be equal to the radius, we can derive a formula for λ;

$$\|p_i - c_d\| \;=\; \sqrt{\lambda^2} = \|p_{d+1} - c_d\| = \sqrt{\sum_{i=1}^{d} \left(\frac{1}{d} - \lambda\right)^2} = \sqrt{d\left(\frac{1}{d} - \lambda\right)^2}.$$

This quadratic equation has the two solutions

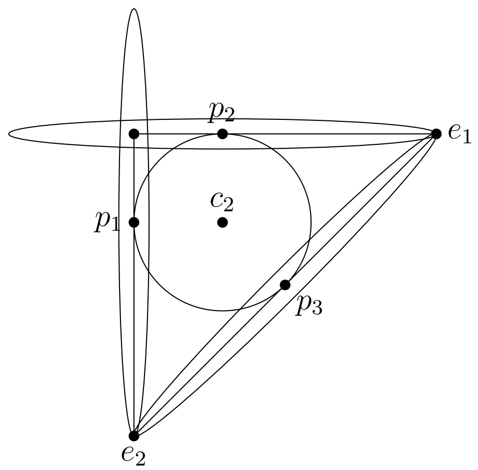
$$\lambda_1 = \frac{\sqrt{d}+1}{\sqrt{d}(d-1)},\ \lambda_2 = \frac{\sqrt{d}-1}{\sqrt{d}(d-1)}.$$

One of the two solutions defines $IS(c_d, r'_d)$ and the other one the circle with center lying on the outside of the d-simplex but still touching face $d+1$ and the extended faces i for $i \in [d]$. Assuming that $d \geq 2$,

$$\begin{aligned} \lambda_2 - \frac{1}{d} &= \frac{d^{\frac{3}{2}} - d - (d-1)\sqrt{d}}{(d-1)d^{\frac{3}{2}}} \\ &= \frac{1-\sqrt{d}}{(d-1)d} \\ &< 0 \end{aligned}$$

holds and we can conclude that $\lambda_2 < \frac{1}{d}$ and therefore $c_d = \lambda_2 I$. The radius r'_d is given as

$$r'_d = \|p_1 - c_d\| = \lambda_2 = \frac{\sqrt{d}-1}{\sqrt{d}(d-1)}.$$



After having defined the point set $P = \{p_1, ..., p_{d+1}\}$, we show how to define the distance functions. For every point p_i, we define an anisotropic distance function

$$d_{p_i}(c) = \sqrt{(p_i - c)^T M_{p_i}(p_i - c)}.$$

For points p_i with $i \in [d]$, M_{p_i} is a diagonal matrix with $(M_{p_i})_{i,i} = \frac{1}{s^2}$ and $\forall j \in [d] \setminus \{i\} :\ (M_{p_i})_{j,j} = \frac{1}{t^2}$.

For the point p_{d+1}, we want to achieve the same scaling but due to the special orientation of the face $d+1$, this is a little bit more difficult. We want to find a rotation matrix R_d and a diagonal matrix D_d so that

$$M_{p_{d+1}} = R_d^T D_d R_d.$$

For the diagonal matrix, we say that $(D_d)_{i,i} = \frac{1}{t^2}$ for all $i \in [d-1]$ and $(D_d)_{d,d} = \frac{1}{s^2}$. Said differently, a point on γI is rotated onto the d'th component; e.g. $R_d I = \sqrt{d} e_d$. Further, all vectors v that are perpendicular to $\lambda_2 I$ fulfill $R_d v = \sum_{i=1}^{d-1} a_i e_i$ for some coefficients a_i. Let us have a look at R_d for small d, make an educated guess for $M_{p_{d+1}}$, and show that this guess is indeed correct.

For $d = 2$, the rotation matrix R_2 is defined as

$$R_2 = \begin{pmatrix} \cos\left(\frac{\pi}{4}\right) & -\sin\left(\frac{\pi}{4}\right) \\ \sin\left(\frac{\pi}{4}\right) & \cos\left(\frac{\pi}{4}\right) \end{pmatrix}.$$

Obviously, $R_2 I = \sqrt{2} e_2$ and $R_2\left(\frac{1}{2}I - e_1\right) = -\frac{e_1}{\sqrt{2}}$. With

$$D_2 = \begin{pmatrix} \frac{1}{t^2} & 0 \\ 0 & \frac{1}{s^2} \end{pmatrix},$$

the anisotropic matrix M_{p_3} is

$$M_{p_3} = \frac{1}{2s^2t^2} \begin{pmatrix} t^2 + s^2 & t^2 - s^2 \\ t^2 - s^2 & t^2 + s^2 \end{pmatrix}.$$

For $d = 3$, the rotation matrix R_3 is

$$R_3 = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}.$$

Again, $R_3 I = \sqrt{3} e_3$ and $R_3\left(\frac{1}{3}I - e_1\right) = -\frac{e_1}{\sqrt{2}} - \frac{e_2}{\sqrt{6}}$. With

$$D_3 = \begin{pmatrix} \frac{1}{t^2} & 0 & 0 \\ 0 & \frac{1}{t^2} & 0 \\ 0 & 0 & \frac{1}{s^2} \end{pmatrix},$$

the matrix M_{p_4} is defined as

$$M_{p_4} = \frac{1}{3s^2t^2} \begin{pmatrix} 2s^2 + t^2 & t^2 - s^2 & t^2 - s^2 \\ t^2 - s^2 & 2s^2 + t^2 & t^2 - s^2 \\ t^2 - s^2 & t^2 - s^2 & 2s^2 + t^2 \end{pmatrix}.$$

Let us give a last example before we make an educated guess for $M_{p_{d+1}}$ and show that this guess is indeed correct. For $d = 4$, the rotation matrix is

$$R_4 = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

$R_4 I = \sqrt{4} e_4$ and $R_4 \left(\frac{1}{4} I - e_1\right) = -\frac{e_1}{\sqrt{2}} - \frac{e_2}{2}$ hold. The diagonal matrix D_4 is

$$D_4 = \begin{pmatrix} \frac{1}{t^2} & 0 & 0 & 0 \\ 0 & \frac{1}{t^2} & 0 & 0 \\ 0 & 0 & \frac{1}{t^2} & 0 \\ 0 & 0 & 0 & \frac{1}{s^2} \end{pmatrix}$$

and it follows that M_{p_5} is defined as

$$M_{p_5} = \frac{1}{4s^2t^2} \begin{pmatrix} 3s^2 + t^2 & t^2 - s^2 & t^2 - s^2 & t^2 - s^2 \\ t^2 - s^2 & 3s^2 + t^2 & t^2 - s^2 & t^2 - s^2 \\ t^2 - s^2 & t^2 - s^2 & 3s^2 + t^2 & t^2 - s^2 \\ t^2 - s^2 & t^2 - s^2 & t^2 - s^2 & 3s^2 + t^2 \end{pmatrix}.$$

Based on those examples, we guess that the matrix $M_{p_{d+1}}$ has

$$(M_{p_{d+1}})_{i,i} = \frac{(d-1)s^2 + t^2}{ds^2t^2}$$

for all $i \in [d]$ and

$$(M_{p_{d+1}})_{i,j} = \frac{t^2 - s^2}{ds^2t^2}$$

for all $i, j \in [d]$ with $i \neq j$. First notice that

$$\|p_{d+1} - c_d\| = \|\frac{1}{d} I - \lambda_2 I\| = \|\frac{\sqrt{d} - 1}{d(d-1)} I\| = \frac{\sqrt{d} - 1}{d(d-1)} \sqrt{d} = \frac{\sqrt{d} - 1}{\sqrt{d}(d-1)}$$

and that

$$\|p_{d+1} - e_i\| = \sqrt{(d-1)\frac{1}{d^2} + \left(\frac{1}{d} - 1\right)^2} = \sqrt{\frac{d-1}{d}}$$

for $i \in [d]$. We claim that

$$d_{p_{d+1}}(c_d) = \frac{\|p_{d+1} - c_d\|}{s}.$$

Using the $M_{p_{d+1}}$ that we have suggested above, we get

$$\begin{aligned} d_{p_{d+1}}(c_d) =& \sqrt{(p_{d+1} - c_d)^T \, M_{p_{d+1}} \, (p_{d+1} - c_d)} \\ =& \sqrt{\left(\frac{\sqrt{d}-1}{d(d-1)}\right)^2 \frac{d^2t^2}{ds^2t^2}} \\ =& \frac{\sqrt{d}-1}{\sqrt{d}(d-1)s} \\ =& \frac{\|p_{d+1} - c_d\|}{s} \end{aligned}$$

as requested and we can conclude that $M_{p_{d+1}}$ scales correctly along the vector I. Next, we show that $M_{p_{d+1}}$ scales correctly in the directions perpendicular to I.

$$\begin{aligned} d_{p_{d+1}}(e_i) =& \sqrt{(p_{d+1} - e_i)^T \, M_{p_{d+1}} \, (p_{d+1} - e_i)} \\ =& \sqrt{(p_{d+1} - e_i)^T \, R} \end{aligned}$$

whereas

$$(R)_i = \frac{1}{d}\frac{dt^2}{ds^2t^2} - \frac{(d-1)s^2 + t^2}{ds^2t^2} = -\frac{(d-1)}{dt^2}$$

and

$$(R)_j = \frac{1}{d}\frac{dt^2}{ds^2t^2} - \frac{t^2 - s^2}{ds^2t^2} = \frac{1}{dt^2}$$

for $j \neq i$. It follows that

$$\begin{aligned} d_{p_{d+1}}(e_i) =& \sqrt{(p_{d+1} - e_i)^T \, R} \\ =& \sqrt{(d-1)\,\frac{1}{dt^2} - \frac{d-1}{dt^2} + \frac{d-1}{dt^2}} \\ =& \frac{\sqrt{d-1}}{\sqrt{d}t} \\ =& \frac{\|p_{d+1} - e_i\|}{t} \end{aligned}$$

and we can conclude that the matrix $M_{p_{d+1}}$ is correct for directions perpendicular to I.

After having defined the point set and the anisotropic distance functions, we are left with analyzing distances between points and centers. So far, we have analyzed the distances $d_{p_{d+1}}(c_d)$ and $d_{p_{d+1}}(e_i)$ and have seen that they are scaled with $\frac{1}{s}$ respectively $\frac{1}{t}$. Next, let us analyze the distances $d_{p_i}(c_d)$, $d_{p_i}(e_j)$, and $d_{p_i}(\mathbf{0})$ for $i, j \in [d]$ and $i \neq j$. For $d_{p_i}(c_d)$, we have

$$\begin{aligned} d_{p_i}(c_d) =& \sqrt{(p_i - c_d)^T M_{p_i} (p_i - c_d)} \\ =& \sqrt{(-\lambda_2 e_i)^T M_{p_i} (-\lambda_2 e_i)} \\ =& \sqrt{(-\lambda_2)^2 \frac{1}{s^2}} \\ =& \frac{\sqrt{d} - 1}{\sqrt{d}(d-1)s}. \end{aligned}$$

and for $d_{p_i}(e_j)$,

$$\begin{aligned} d_{p_i}(e_j) =& \sqrt{(p_i - e_j)^T M_{p_i} (p_i - e_j)} \\ =& \sqrt{\frac{d-2}{t^2} \left(\frac{\sqrt{d} - 1}{\sqrt{d}(d-1)} \right)^2 + \frac{1}{t^2} \left(\frac{\sqrt{d} - 1}{\sqrt{d}(d-1)} - 1 \right)^2} \\ =& \frac{\sqrt{d-1}}{\sqrt{d}t}. \end{aligned}$$

Finally, for $d_{p_i}(\mathbf{0})$,

$$\begin{aligned} d_{p_i}(\mathbf{0}) =& \sqrt{(p_i - \mathbf{0})^T M_{p_i} (p_i - \mathbf{0})} \\ =& \sqrt{(d-1) \left(\frac{\sqrt{d} - 1}{\sqrt{d}(d-1)} \right)^2 \frac{1}{t^2}} \\ =& \frac{\sqrt{d} - 1}{\sqrt{d(d-1)}t}. \end{aligned}$$

Observe that

$$d_{p_i}(e_i) - d_{p_i}(\mathbf{0}) = \frac{\sqrt{d-1}}{\sqrt{d}t} - \frac{\sqrt{d} - 1}{\sqrt{d(d-1)}t} = \frac{\sqrt{d} - 1}{\sqrt{d-1}t} > 0 \qquad (28)$$

for all $d \geq 2$. Notice further that for all p_i with $i \in [d+1]$,

$$d_{p_i}(c_d) = \frac{\sqrt{d} - 1}{\sqrt{d}(d-1)s}$$

and we can conclude that $MB(P)$ has radius

$$r_d = \frac{\sqrt{d}-1}{\sqrt{d}(d-1)s}.$$

Removing p_{d+1} form the point set yields a smallest enclosing ball $MB(P \setminus \{p_{d+1}\})$ with radius

$$r_{d-1,p_{d+1}} \leq \frac{\sqrt{d}-1}{\sqrt{d(d-1)}t}$$

as with $\mathbf{0}$, we have a witness for a center. If we remove a point p_i with $i \in [d]$, we know that all points in $P \setminus \{p_i\}$ have distance

$$\frac{\sqrt{d-1}}{\sqrt{d}t}$$

to e_i and we can conclude that

$$r_{d-1,p_i} \leq \frac{\sqrt{d-1}}{\sqrt{d}t}.$$

With Equation 28 it follows that for all $i \in [d+1]$,

$$r_{d-1,p_i} \leq \frac{\sqrt{d-1}}{\sqrt{d}t}$$

holds. Choosing

$$s \ll \frac{\sqrt{d}-1}{(d-1)^{\frac{3}{2}}}t$$

lets us conclude that

$$r_{d-1,p_i} \leq \frac{\sqrt{d-1}}{\sqrt{d}t} \ll \frac{\sqrt{d}-1}{\sqrt{d}(d-1)s} = r_d$$

and it follows that the decrease in radius can be arbitrarily large. We can conclude that removing an arbitrary point from the point set $P \subset \mathbb{R}^d$ gives us a point set with an arbitrarily smaller smallest enclosing ball $MB(P \setminus \{p\})$ compared to the smallest enclosing ball $MB(P)$.

So far, we have shown that the decrease in radius can be arbitrarily large for an SC point set. Next, let us show that an arbitrary decrease in radius implies that there cannot be ϵ-core sets for points having distance functions with strictly convex level sets. To do so, we first have to compute the distances between c_{d-1,p_i} and p_i. As we do not have the exact location of c_{d-1,p_i}, we have to approximate and assume that $s \lll t$ and that therefore $c_{d-1,p_i} \approx e_i$ for $i \in [d]$ or $c_{d-1,p_{d+1}} \approx \mathbf{0}$. After having gone through the calculations for $d_{p_i}(e_i)$ and $d_{p_{d+1}}(\mathbf{0})$, we will show that the chosen approximations do *not* destroy the result.

For $i \in [d]$,

$$\begin{aligned}
d_{p_i}(c_{d-1,p_i}) \approx & d_{p_i}(e_i) \\
= & \sqrt{(p_i - e_i)^T M_{p_i} (p_i - e_i)} \\
= & \sqrt{\left(\left(\sum_{j\in[d]\setminus\{i\}} \lambda_2 e_j\right) - e_i\right)^T M_{p_i} \left(\left(\sum_{j\in[d]\setminus\{i\}} \lambda_2 e_j\right) - e_i\right)} \\
= & \sqrt{\left(\left(\sum_{j\in[d]\setminus\{i\}} \lambda_2 e_j\right) - e_i\right)^T R}
\end{aligned}$$

whereas

$$\begin{aligned}
R_i = & (d-1)\lambda_2 \frac{t^2 - s^2}{ds^2t^2} - \frac{(d-1)s^2 + t^2}{ds^2t^2} \\
= & \frac{t^2 + s^2\left(d^{\frac{3}{2}} - 1\right)}{d^{\frac{3}{2}} s^2 t^2}
\end{aligned}$$

and

$$\begin{aligned}
R_j = & ((d-2)\lambda_2 - 1)\frac{t^2 - s^2}{ds^2t^2} + \lambda_2 \frac{(d-1)s^2 + t^2}{ds^2t^2} \\
= & \frac{t^2(1-d) + s^2(d^{\frac{3}{2}} - 1)}{d^{\frac{3}{2}}(d-1)s^2t^2}
\end{aligned}$$

for $j \in [d] \setminus \{i\}$. Therefore, it follows that

$$\begin{aligned}
d_{p_i}(c_{d-1,p_i}) \approx & \sqrt{\left(\left(\sum_{j\in[d]\setminus\{i\}} \lambda_2 e_j\right) - e_i\right)^T R} \\
= & \sqrt{(d-1)\lambda_2 R_j - R_i} \\
= & \sqrt{\frac{t^2(d-1) + s^2\left(d^3 - 2d^{\frac{3}{2}} + 1\right)}{d^2(d-1)s^2t^2}}.
\end{aligned} \tag{29}$$

For $d_{p_{d+1}}(c_{d-1,p_{d+1}})$, we have

$$\begin{aligned} d_{p_{d+1}}(c_{d-1,p_{d+1}}) \approx & d_{p_{d+1}}(\mathbf{0}) \\ = & \sqrt{(p_{d+1} - \mathbf{0})^T M_{p_{d+1}} (p_{d+1} - \mathbf{0})} \\ = & \sqrt{\frac{1}{d} I^T M_{p_{d+1}} \frac{1}{d} I} \\ = & \sqrt{\frac{t^2}{ds^2 t^2}} \\ = & \frac{1}{\sqrt{d}s} \end{aligned}$$

We can assume that we choose s and t so that $st = 1$. Let us argue that our approximation of c_{d-1,p_i} does *not* destroy the result. The distance between the center c_{d-1,p_i} and e_i respectively between $c_{d-1,p_{d+1}}$ and $\mathbf{0}$ is dependent on s and t. For a given dimension d, we choose s and t so that $\|c_{d-1,p_i} - e_i\| \leq \frac{1}{2d}$ and $\|c_{d-1,p_{d+1}} - \mathbf{0}\| \leq \frac{1}{2\sqrt{d}}$. It follows that the distance between c_{d-1,p_i} and e_i as perceived by p_i is at most $\frac{1}{2ds} = \frac{t}{2d}$ and we can conclude that $d_{p_i}(c_{d-1,p_i}) = O(t)$. Similarly, the distance between $c_{d-1,p_{d+1}}$ and $\mathbf{0}$ as perceived by p_{d+1} is at most $\frac{1}{2\sqrt{d}s} = \frac{t}{2\sqrt{d}}$ and therefore $d_{p_{d+1}}(c_{d-1,p_{d+1}}) = O(t)$.

As $d - 1 > 0$ for all $d \geq 2$, we can conclude that we can choose $s \ll t$ so that $d_{p_i}(c_{d-1,p_i})$ (see Equation 29) is arbitrarily large for all $i \in [d+1]$. From

$$r_{d-1,p_i} \leq \frac{\sqrt{d-1}}{\sqrt{d}t},$$

it follows immediately that ϵ has to be arbitrarily large to fulfill

$$d_{p_i}(c_{d-1,p_i}) \leq (1+\epsilon) r_{d-1,p_i}.$$

It follows that ϵ has to be arbitrarily large even for a core set of size d and we conclude that there cannot be a core set of size $f(\epsilon)$ with f independent on $|P|$ and d.

Even with this negative result about ϵ-core sets, we want to mention that this does *not* imply that c_{d-1,p_i} is a bad approximation for the smallest enclosing ball $MB(P)$. Recalling Definition 7.1 for $(1+\epsilon)$ approximation,

$$d_{p_i}(c_{d-1,p_i}) \leq (1+\epsilon) r_d$$

has to be fulfilled. As

$$r_d = \frac{\sqrt{d} - 1}{\sqrt{d}(d-1)s},$$

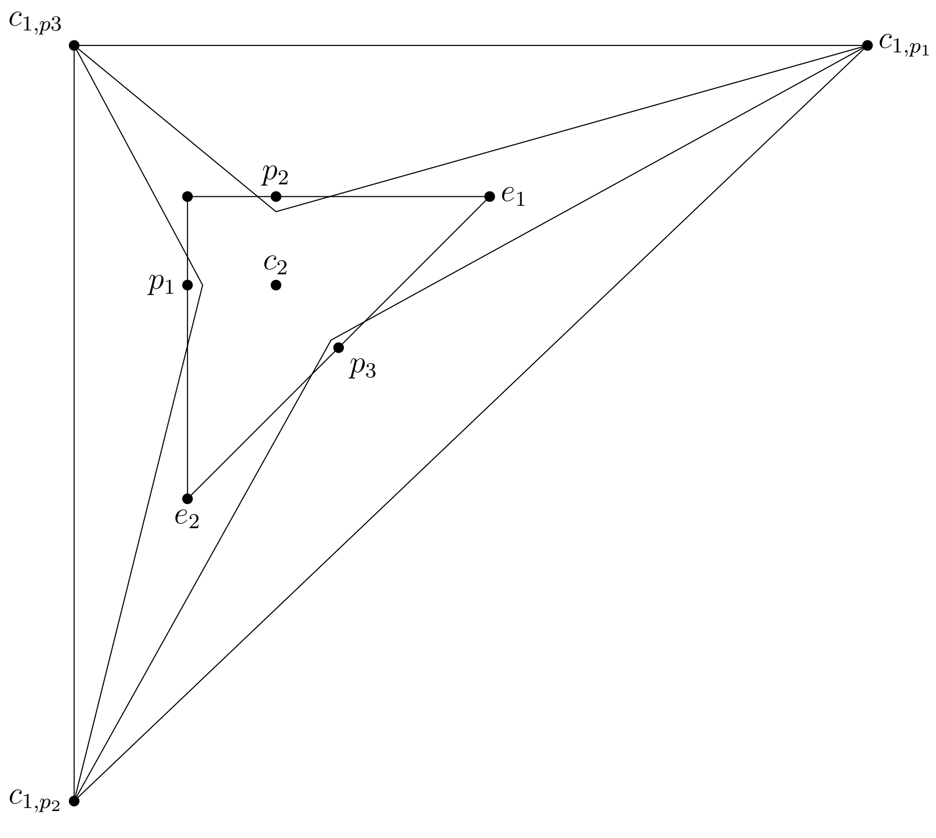


Figure 20: Sketch showing that the center c_{1,p_i} can be moved arbitrarily far from c_d. Notice that we do not shown the level set of the distance functions itself but rather approximations that are *not* strictly convex.

$s = \frac{1}{t}$, and

$$d_{p_i}(c_{d-1,p_i}) \leq \max\{\frac{1}{\sqrt{d}s}, \sqrt{\frac{t^2\,(d-1) + s^2\left(d^3 - 2d^{\frac{3}{2}} + 1\right)}{d^2(d-1)s^2t^2}}\},$$

it is easy to see that both $d_{p_i}(c_{d-1,p_i})$ and r_d grow at about the same rate with t and that it might therefore be possible to find an ϵ independent on t fulfilling

$$d_{p_i}(c_{d-1,p_i}) \leq (1+\epsilon)r_d.$$

To weaken this observation, notice that this only holds because the location of c_{d-1,p_i} is restricted to the d-simplex formed by $\mathbf{0} \cup \bigcup_{i\in[d]} e_i$ which is an implication of the symmetry of the involved distance functions. Choosing distance functions that are asymmetrical will allow to place c_{d-1,p_i} arbitrarily far from c_d and at the same time make $d_{p_i}(c_{d-1,p_i})$ arbitrarily large. We do not proceed formally as this is likely to get messy but consult Figure 20 for an informal description.

Observe that in case of symmetrical distance functions, it seems that the distance between c_d and $c_{d-1,p}$ for an optimal point p cannot be arbitrarily large. With above observation about a similar rate of growth of $d_{p_i}(c_{d-1,p_i})$ and r_d, this seems to hint at the possibility that an approximation algorithm for anisotropic point sets might exist.

7.3 Maximum Decrease in Radius with Dependence on the Distance Function for SC Point Sets

With the results from the previous section, we know that it is *not* possible to bound the decrease in radius for SC point sets by a constant factor. What we try to do here is to show that the decrease in radius is bounded if we allow the bound to be dependent on s and t as presented below. The motivation for doing so is the intuition that the decrease in radius and core sets are *somehow related concepts*.

In Section 7.1, we tried to prove the existence of ϵ-core sets based on the fact that the decrease in radius is bounded by a constant factor but unfortunately, we failed doing so. If one is succeeding to show this relation for L_2, one might be able to proceed similarly for SC point sets and show that for SC point sets, there are core sets of size $\epsilon f(s,t)$ that are $(1+\epsilon)$-approximations.

Right now, we are not able to make the formal connection between the decrease in radius and the existence of core sets but we state the results for

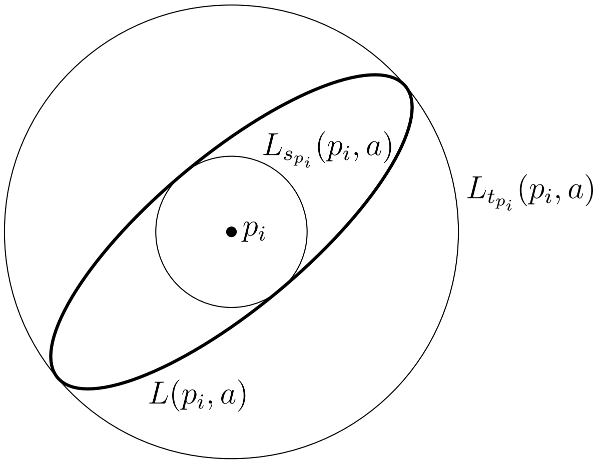


Figure 21: The level set $L(p_i, a)$ (thick) with the associated level sets $L_{s_{p_i}}(p_i, a)$ and $L_{t_{p_i}}(p_i, a)$.

completeness and for the case that the formal connection between decrease in radius and core sets is shown.

Assume that we have a point set P with arbitrary distance functions that have strictly convex level sets. For all $p \in P$ and $a \in [0, \infty)$, let $L(p, a) = \{x \mid x \in \mathbb{R}^d,\ d_p(x) \leq a\}$ and assume that

$$L_{s_p}(p, a) = \{x \mid x \in \mathbb{R}^d,\ d_{p,s_p}(x) = \frac{\|p - x\|}{s_p} \leq a\} \subseteq L(p, a)$$

with s_p fulfilling above condition and being as large as possible. Analogously, we assume that for all $p \in P$ and $a \in [0, \infty)$,

$$L(p, a) \subseteq L_{t_p}(p, a) = \{x \mid x \in \mathbb{R}^d,\ d_{p,t_p}(x) = \frac{\|p - x\|}{t_p} \leq a\}$$

with t_p being as small as possible while still fulfilling above condition.

Define $s = \min_{p \in P}\{s_p\}$ and $t = \max_{p \in P}\{t_p\}$ and observe that $s \leq t$. Let us say that the point set P_s has the same points as P but that the points use the distance functions $d_p(x) = \frac{\|p-x\|}{s}$; the point set P_t is defined analogously.

Lemma 7.10. *Assume that the smallest enclosing ball $MB(P)$ has radius r_d. A point set P_s as defined above has a smallest enclosing ball $MB(P_s)$ with radius $r_{d,s}$ and*

$$r_{d,s} \geq r_d$$

holds.

Proof. Assume not and it follows that there has to be a center $c_{d,s}$ with $d_{p,s}(c_{d,s}) \leq r_{d,s} < r_d$ for all $p \in P_s$. With $s = \min_{p \in P}\{s_p\}$,

$$d_{p,s_p}(c_{d,s}) = \frac{\|p - c_{d,s}\|}{s_p} \leq \frac{\|p - c_{d,s}\|}{s} = d_{p,s}(c_{d,s})$$

holds for all $p \in P_s$ and it follows that

$$c_{d,s} \in L_{s_p}(p, r_{d,s})$$

for all $p \in P_s$. As $L_{s_p}(p, r_{d,s}) \subseteq L(p, r_{d,s})$ holds for all $p \in P$, we can conclude that we have found a smallest enclosing ball with radius $r_{d,s} < r_d$ for the point set P; a contradiction to our assumption that the radius of the smallest enclosing ball $MB(P)$ is r_d. □

As we want to bound

$$\min_{p \in P} \frac{r_d}{r_{d-1,p}} \tag{30}$$

from above, we have to find a point set that maximizes Equation 30. Using the same argumentation as in the proof of Lemma 7.4, we can conclude that $|P| \leq d + 1$ has to hold.

Say that a point $q \in P$ is optimal if $MB(P \setminus \{q\})$ has radius r_{d-1} so that $\frac{r_d}{r_{d-1}}$ is minimized. Let us have a look at the point set $Q = P \setminus \{q\}$ for an optimal q and assume that the smallest enclosing ball $MB(Q)$ has center $c_{d-1,q}$ and radius $r_{d-1,q}$. Say that the point set Q_t has the same points as Q but the points use the distance function $d_p(x) = \frac{\|p-x\|}{t}$.

Lemma 7.11. *Assume that the smallest enclosing ball $MB(Q)$ has radius $r_{d-1,p}$ and center $c_{d-1,p}$. A point set Q_t as defined above has a smallest enclosing ball $MB(Q_t)$ with radius $r_{d-1,p,t}$ and*

$$r_{d-1,p,t} \leq r_{d-1,p}$$

holds.

Proof. Observe that for all $q \in Q$, $c_{d-1,p} \in L(q, r_{d-1,p})$. With

$$L(q, r_{d-1,p}) \subseteq L_{t_p}(q, r_{d-1,p}) \subseteq L_t(q, r_{d-1,p}),$$

this implies that $c_{d-1,p} \in L_t(q, r_{d-1,p})$. It follows that $d_q(c_{d-1,p}) \leq r_{d-1,p}$ for all $q \in Q$ and therefore $r_{d-1,p,t} \leq r_{d-1,p}$. □

With Lemmata 7.10 and 7.11, we can conclude that

$$\min_{p\in P}\frac{r_d}{r_{d-1,p}} \le \min_{p\in P_t}\frac{r_{d,s}}{r_{d-1,p,t}}.$$

In a next step, we show that we can bound

$$\min_{p\in P_t}\frac{r_{d,s}}{r_{d-1,p,t}}$$

from above by moving the point set P into equilateral position; this is possible as $|P| \le d+1$. Observe that all points of Q_t are contained in P_s but that the points in the two point sets use different distance functions.

Lemma 7.12. *Assume that P_s and Q_t as defined above. Let P'_s be a point set in equilateral position and let $d_p(c) = \frac{\|p-c\|}{s}$ for all $p \in P'_s$. Further, $Q'_t = P'_s \setminus \{p\}$ for a $p \in P'_s$ and for all $q \in Q'_t$, $d_q(c) = \frac{\|q-c\|}{t}$. Say that $MB(P'_s)$ has center $c'_{d,s}$ and radius $r'_{d,s}$ and that $MB(Q'_t)$ has center $c'_{d-1,p,t}$ and radius $r'_{d-1,p,t}$. Then*

$$\min_{p\in P_t}\frac{r_{d,s}}{r_{d-1,p,t}} \le \min_{p\in P'_t}\frac{r'_{d,s}}{r'_{d-1,p,t}}$$

holds.

Proof. From Lemma 7.4, we know that for point sets with L_2 distance function,

$$\min_{p\in P'_1}\frac{r_d}{r_{d-1,p}}$$

is maximized by a point set P'_1 in equilateral position. Unfortunately, the point sets P_s and Q_t have not L_2 distance functions but rather *scaled* L_2 distance functions. As all points in P_s use the same scale factor, $MB(P_1)$ and $MB(P_s)$ share the center $c_{d,s}$ and $r_{d,1} = sr_{d,s}$ with $r_{d,1}$ being the radius of $MB(P_1)$. Analogously $MB(Q_1)$ and $MB(Q_t)$ share the center $c_{d-1,p,t}$ and $r_{d-1,p,1} = tr_{d-1,p,t}$ with $r_{d-1,p,1}$ the radius of $MB(Q_1)$.

For P_1, we know that a point set P'_1 in equilateral position fulfills

$$\min_{p\in P_1}\frac{r_{d,1}}{r_{d-1,p,1}} \le \min_{p\in P'_1}\frac{r'_{d,1}}{r'_{d-1,p,1}}$$

and this allows us to conclude that

$$\min_{p\in P_t}\frac{r_{d,s}}{r_{d-1,p,t}} = \min_{p\in P_1}\frac{tr_{d,1}}{sr_{d-1,p,1}} \le \min_{p\in P'_1}\frac{tr'_{d,1}}{sr'_{d-1,p,1}} = \min_{p\in P'_t}\frac{r'_{d,s}}{r'_{d-1,p,t}}$$

holds and our claim follows. $\square$

It remains to analyze

$$\min_{p \in P'_t} \frac{r'_{d,s}}{r'_{d-1,p,t}}$$

in order to bound the maximal decrease in radius from above.

Lemma 7.13. *Let P be an SC point set, $MB(P)$ have radius r_d, and $MB(P \setminus \{p\})$ radius $r_{d-1,p}$. Then*

$$\min_{p \in P} \frac{r_d}{r_{d-1,p}} \leq \frac{2t}{\sqrt{3}s}.$$

Proof. With Lemma 7.12, we know that

$$\min_{p \in P} \frac{r_d}{r_{d-1,p}} \leq \min_{p \in P_t} \frac{r_{d,s}}{r_{d-1,p,t}} \leq \min_{p \in P'_t} \frac{r'_{d,s}}{r'_{d-1,p,t}} = \min_{p \in P'_1} \frac{tr'_{d,1}}{sr'_{d-1,p,1}}.$$

It remains therefore to find a formula for

$$\min_{p \in P'_1} \frac{tr'_{d,1}}{sr'_{d-1,p,1}}.$$

From Lemma 7.3, we know that $MB(P'_1)$ has radius

$$r'_{d,1} = \sqrt{\frac{d}{2(d+1)}}u$$

with u being the distance between two points in P'_1. As Q'_1 is a regular $(d-1)$-simplex,

$$r'_{d-1,p,1} = \sqrt{\frac{d-1}{2d}}u.$$

It follows that

$$\min_{p \in P'_t} \frac{r'_{d,s}}{r'_{d-1,p,t}} = \frac{td}{s\sqrt{(d+1)(d-1)}} = f(d,s,t).$$

Differentiating $f(d,s,t)$ yields

$$\frac{\partial}{\partial d} \frac{td}{s\sqrt{(d+1)(d-1)}} = -\frac{t}{s(d-1)^{\frac{3}{2}}(d+1)^{\frac{3}{2}}}$$

and for $s, t > 0$ and $d \geq 2$, this is always negative. It follows that

$$\min_{p \in P} \frac{r_d}{r_{d-1,p}} \leq \frac{2t}{\sqrt{3}s}$$

for all $d \geq 2$. □

We can conclude that the decrease in radius is bounded from above by a function that is solely dependent on s and t.

7.4 Potential Core Set Algorithms

The results from the previous section seem to hint that there are core sets with size dependent on ϵ, s, and t. It is an obvious guess that one of the core set algorithms that work for L_2 might as well work for SC point sets. We list two core set algorithms and show for the first one that it does not work for SC point sets. For the second algorithm, we conjecture that it works for SC point sets but we are not able to prove this.

Bădoiu and Clarkson present in [5] a simple algorithm that produces an ϵ-core set in $\frac{1}{\epsilon^2}$ steps:

procedure `core-set1(P)`
begin
  Select an arbitrary point $p \in P$.
  Set $c_0 = p$.
  for $i = 0$; $i < \frac{1}{\epsilon^2}$; $i{+}{+}$ **do**
    Select the point $p_i \in P$ so that $d_{p_i}(c_i)$ is maximal.
    Set $c_{i+1} = c_i + \frac{p - c_i}{i+1}$.
  end
  return $c_{\frac{1}{\epsilon^2}}$
end

Algorithm 3: First algorithm by Bădoiu and Clarkson.

The counter example showing that Algorithm 3 does *not* work for SC point sets is a point set that lies on a line and that has the center *far away* from this line; see Figure 22 for an example. No matter how many steps we make for such a configuration, the center c_i always stays on the line defined by the point set. This means that independent on the size of the core set, we can find a point set with c_i being a *bad* approximation.

The second algorithm as presented in [5] is defined as follows:

procedure `core-set2(P)`
`// Let` $MB(S_i)$ `have center` c_i `and radius` r_i`.`
begin
  Select an arbitrary point $p \in P$.
  Set $S_0 = \{p\}$.
  for $i = 0$; $i < \frac{2}{\epsilon}$; $i{+}{+}$ **do**
    Select the point $p_i \in P$ so that $d_{p_i}(c_i)$ is maximal.
    Set $S_{i+1} = S_i \cup p_i$.
  end
  return $MB(S_{\frac{2}{\epsilon}})$
end

Algorithm 4: Second algorithm by Bădoiu and Clarkson.

We cannot find a counter example showing that Algorithm 4 does not

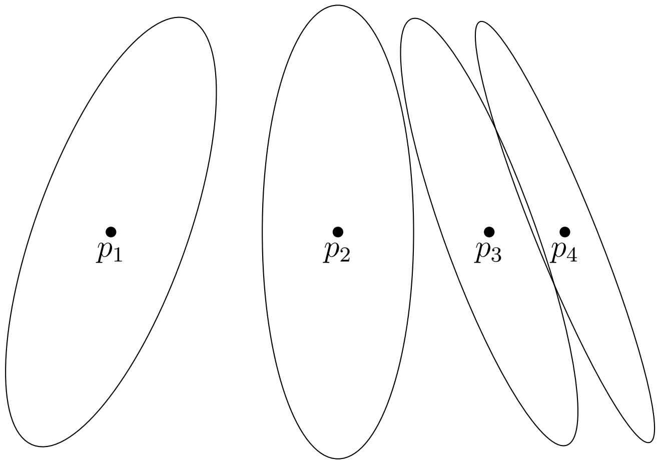


Figure 22: The point set $P = \{p_1, p_2, p_3, p_4\}$ lying on a line is a counter example showing that the first algorithm as presented by Bădoiu and Clarkson is not working for SC point sets.

work for SC point sets. We assume that Algorithm 4 works for SC point sets if we repeat the last step $\frac{2}{\epsilon} f(s,t)$ times with $f(s,t)$ being a function that is dependent on s and t. Unfortunately, we are not able to prove that Algorithm 4 indeed produces core sets for SC point sets. Due to the different geometric setting that we are facing for SC point sets, the proof strategy as used in [5] cannot be applied for SC point sets; e.g. Lemma 2.1 from [5] does not hold for SC point sets.

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